Assignment 2

Due Sept 30 2026

Out of 11 marks
The ones graded are marked with *

1. "running time is at least $O(n^2)$" is meaningless.

1

True because Big O only indicates upper bound.
The statement is like saying, "The algorithm takes at least no more than n^2 time."

2. Prove "Recip Rule"

4

$f(n) \in O(g(n))$

$\Rightarrow \exists\ c>0,\ n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0$

$\Rightarrow$ by multiplying both sides by $\frac{1}{f(n)\cdot g(n)}$, we have that

for those values of c, n_0: $\frac{1}{g(n)} \leq c \cdot \frac{1}{f(n)}\ \forall n \geq n_0$.

$\Rightarrow \frac{1}{g(n)} \in O\left(\frac{1}{f(n)}\right)$

$\therefore f(n) \in O(g(n)) \Rightarrow \frac{1}{g(n)} \in O\left(\frac{1}{f(n)}\right)$ ▨

3. Show that $3n^2 + 4n - \frac{\log n}{\log \log n} \in O(n^2)$ using the Definition.

4

$$3n^2 + 4n - \frac{\log n}{\log \log n} \leq 3n^2 + 4n \qquad \forall n \geq 1$$
$$\leq 3n^2 + 4n^2 \qquad \forall n \geq 1$$
$$\leq 7n^2 \qquad \forall n \geq 1$$

$\therefore 3n^2 + 4n - \frac{\log n}{\log \log n} \in O(n^2)$ demoed by $n_0 = 1,\ c = 7$. ▨

*4 Prove that $6n \lg n + 4n^3 - 8n \in \Theta(n^3)$.

4

I will use the rules.

① $\lg n \in O(n)$ Log domination rule

② $6n \in O(n^2)$ polynomial rule

③ $6n \lg n \in O(n^3)$ 1, 2 Product Rule

④ $4n^3 - 8n \in O(n^3)$ polynomial rule

⑤ $6n \lg n + 4n^3 - 8n \in O(n^3)$ ③, ④ Sum Rule ▨

Now to show $n^3 \in O(6n \lg n + 4n^3 - 8n)$.

$n^2 \leq 4n^2 - 8 \quad \forall n$ where $3n^2 \geq 8$ i.e. when $n^2 \geq \frac{8}{3}$, so true when $n \geq 3$

$\therefore n^3 \leq 4n^3 - 8n \quad \forall n \geq 3$

$\therefore n^3 \leq 4n^3 - 8n + 6 \lg n \quad \forall n \geq 3$, since $6 \lg n$ is positive in this range

$\therefore n^3 \in O(6 \lg n + 4n^3 - 8)$, demoed by $n_0 = 3$ and $c = 1$.

Since $n^3 \in O(6 \lg n + 4n^3 - 8)$ and $6 \lg n + 4n^3 - 8 \in O(n^3)$,

$6 \lg n + 4n^3 - 8 \in \Theta(n^3)$,

* 5. Is $\frac{n}{\log n} \in O(n)$? in $\Omega(n)$? (No (2)) in $\Theta(n)$? (No. (3)) Y or N and prove.

We did this in class

$\frac{n}{\log n} \in O(n)$. Yes. (1)

Proof. Claim: $1 \in O(\lg n)$

Proof: $1 \leq \lg n \quad \forall n \geq 2$

$\therefore 1 \in O(\lg n)$ demoed by $n_0 = 2$, $c = 1$. ▨

Now we prove the main theorem.

① $n \in O(n)$ Removal of constant factors

② $1 \in O(\lg n)$ claim proved above.

③ $\frac{1}{\lg n} \in O(1)$ ②, Recip rule.

④ $\frac{n}{\lg n} \in O(n)$ ①, ④ Product Rule. ▨ (1)

We now have to prove that $\frac{n}{\lg n} \notin \Omega(n)$.

Proof: BWOC. $\frac{n}{\lg n} \in \Omega(n)$.

Then $n \in O(\frac{n}{\lg n})$

ie $\exists c, n_0$ s.t. $n \leq c \cdot \frac{n}{\lg n} \quad \forall n \geq n_0$

$\Rightarrow \lg n \leq c \quad \forall n \geq n_0$

$\Rightarrow n \leq 2^c \quad \forall n \geq n_0$.

But let n be $\max(n_0, 2^c + 1)$.

Then $n > 2^c$

$\Rightarrow \Leftarrow$

$\therefore\ n \notin O(\frac{n}{\lg n})$, and $\frac{n}{\lg n} \notin \Omega(n)$. ✓ (2)

Since $\frac{n}{\lg n} \notin \Omega(n)$, $\frac{n}{\lg n} \notin \Theta(n)$. ✓ (3)

4 *6. Order the given functions, top grows slowest, bottom fastest.

$n + 10\sqrt{n}$

$10n \lg\lg n$

$n^2 \lg n$

2^n

$(2.3)^n n^2$

$n!$

n^n, $2^{n \lg n}$ (because $2^{\lg n \cdot n} = (2^{\lg n})^n = n^n$)

7. When size of input doubles...

a) $f(n) = n$ running time doubles eg. $100 \to 200$ ✓2

b) $f(n) = n^2$ running time quadruples eg $100 \to 400$ ✓2

c) $f(n) = 2^n$ running time squares eg $100 \to 10000$ ✓2

8. *a) $T(n) = 3T(\frac{n}{4}) + n$ Note: $\log_4 3 = .79$

$n \in ?$ $O(n^{.79-\varepsilon})$ for some $\varepsilon > 0$

$\Theta(n^{.79})$

$\Omega(n^{.79+\varepsilon})$ "

3

$n \in \Omega(n^{.79+\varepsilon})$ ie could use $\varepsilon = .001$

We must check regularity:

$\exists ?\ c$ such that $3(\frac{n}{4}) \le cn$ Yes $c = \frac{3}{4} + \varepsilon$ will do.
$1 < c < 0$

$\therefore\ T(n) \in \Theta(n)$.

b) $T(n) = 4T(\frac{n}{2}) + n^2$

3

$O(n^{2-\varepsilon})$ for some $\varepsilon > 0$?

$n^2 \in ?$ → $\Theta(n^2)$

$\Omega(n^{2+\varepsilon})$ ✓

$\therefore\ T(n) \in \Theta(n^2 \lg n)$

c) $T(n) = 6T(n/3) + n \log n$ $\quad \log_3 6 \approx 1.5$

3

$n \log n \overset{?}{\in}$ → $O(n^{1.5-\varepsilon})$

$\Theta(n^{1.5})$

$\Omega(n^{1.5+\varepsilon})$

$\therefore\ T(n) \in \Theta(n^{\log_3 6})$