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# O(n lg n) Algorithm for Closest Pair

(Computational Geometry)

Given: n points on a plane

Find: the pair of points that are closest to one another

i.e. minimize

$$d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

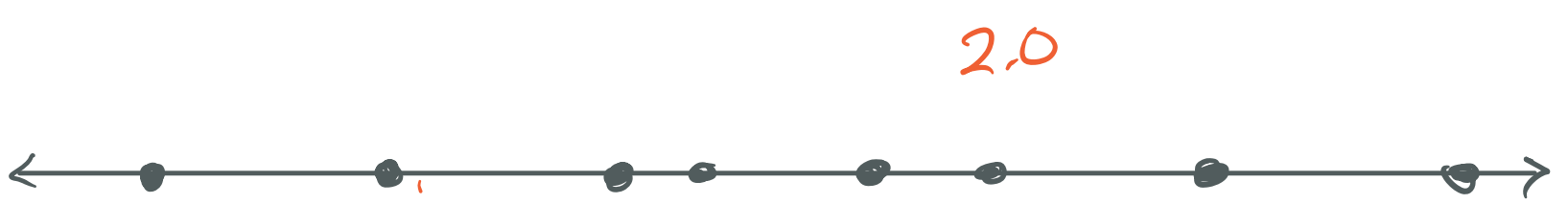
where $p_1 = (x_1, y_1)$

$p_2 = (x_2, y_2)$

are some pair of distinct points in the given set.

{ 7.6, 3.2, 11.9, 2.0, 6.6 }

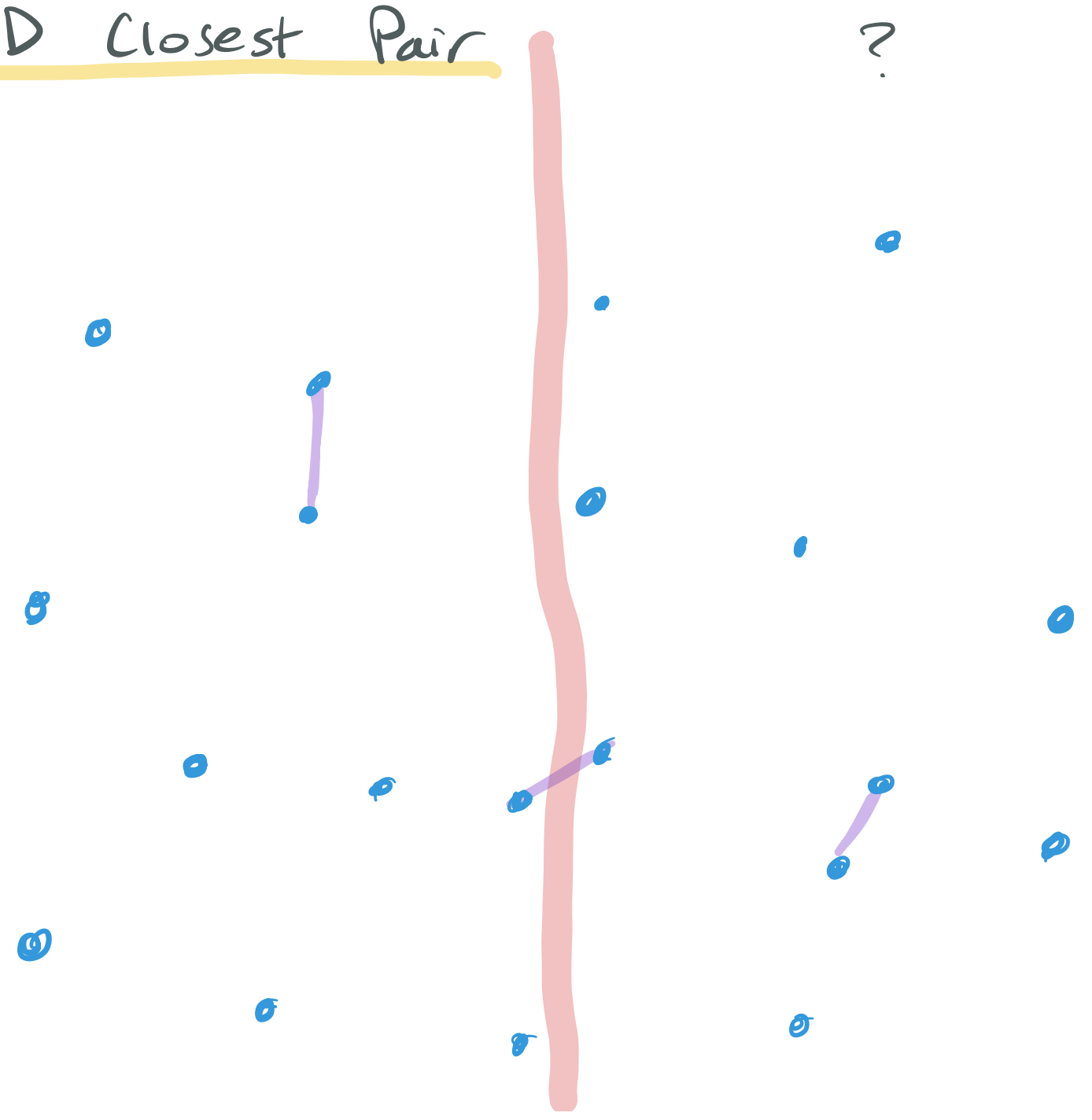
What if 1-dimensional ?



## 1D_Closest_Pair  algorithm sketch

- sort the points
- for each point, calculate distance from previous point in sorted order
  - keep track of minimum value calculated and the point that yielded it.
- return the overall minimum.

## 2D Closest Pair ?



Want to find closest pair in time $O(n \lg n)$

Can sort the points in the $x$-dimension in $O(n \log n)$

" " " $y$-dimension in $O(n \log n)$

We can do each of these steps as part of our preprocessing "for free" if it helps us.

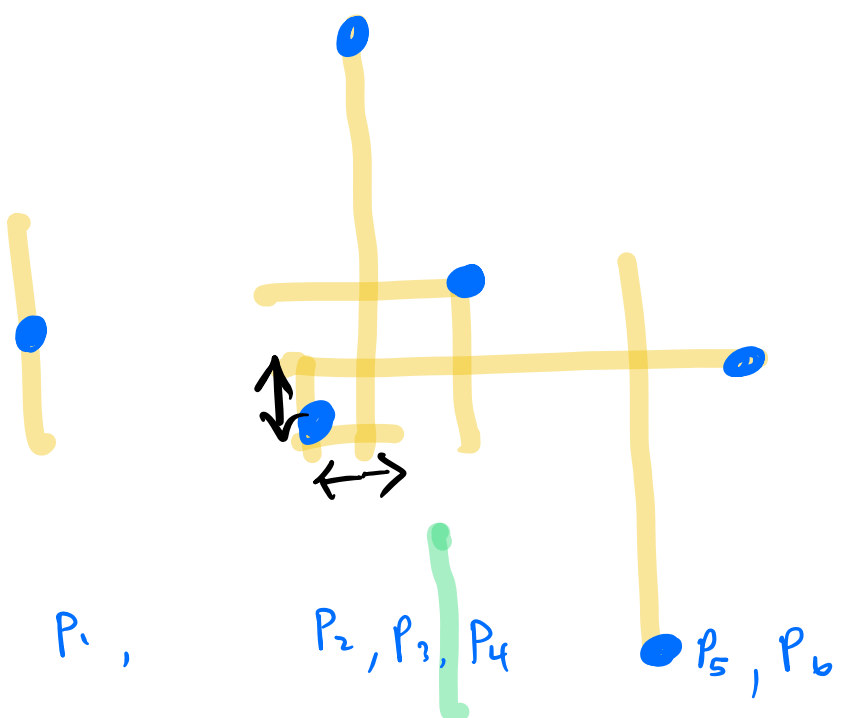
Why do we consider it "free" to do this work?

- if we have $O(n \log n)$ as our goal

then a constant number of calls to $O(n \log n)$-time subroutines does not increase the asymptotic running time

But....

the closest pair may not be closest in x-direction nor closest in y-direction



## Brute Force approach:

- $n^2$ pairs of points: compute the distance in each case and note the minimum.

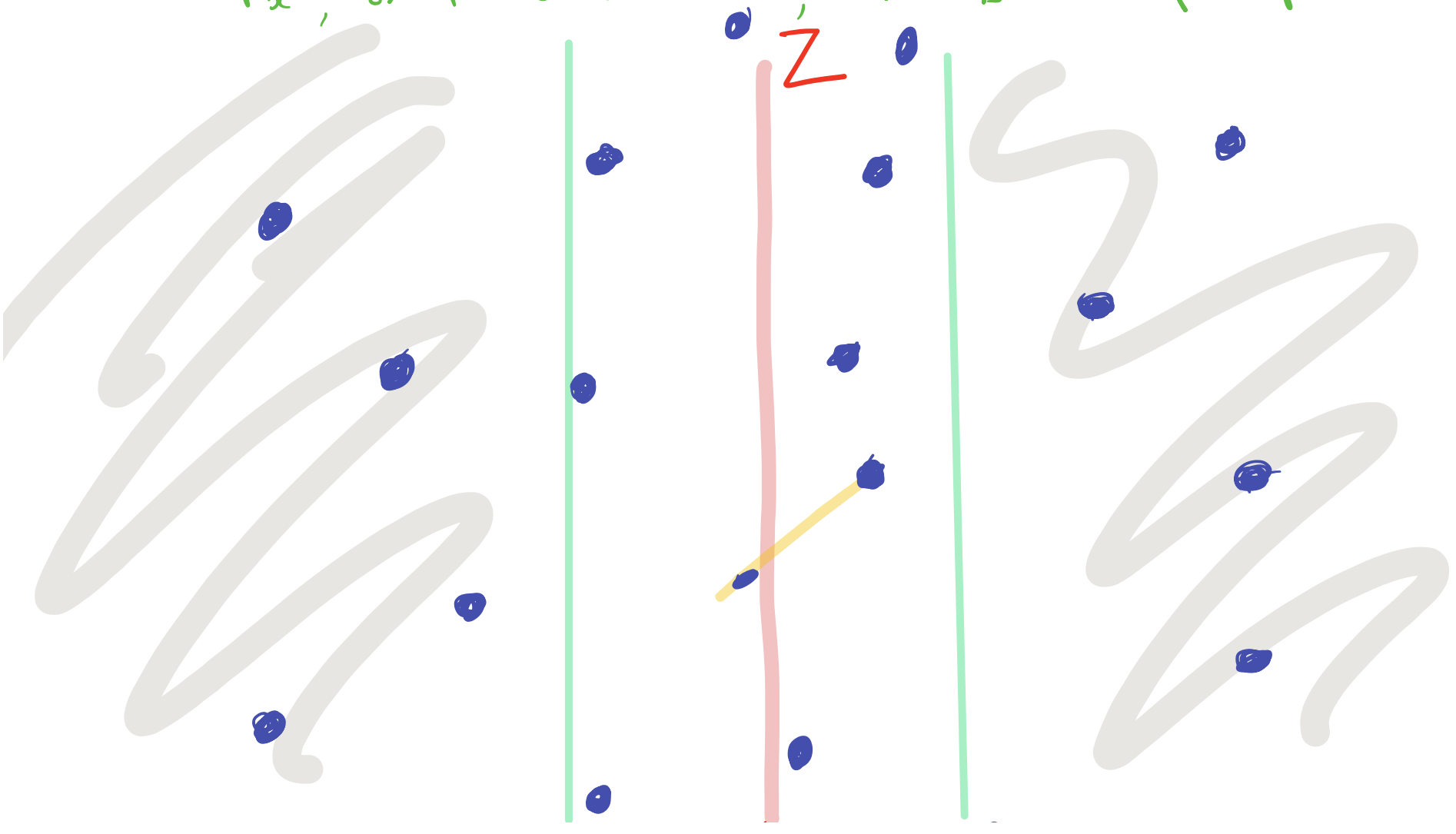
## Divide & Conquer Approach

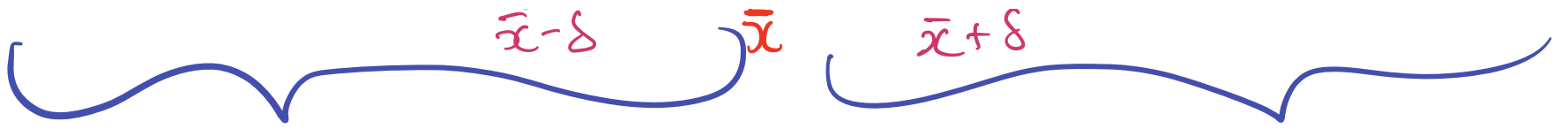
$P_x$ = list of points, sorted by x-coord

$P_y$ = list of points, sorted by y-coord

$P_z$ = $P_y$ filtered so x-vals are $\in [\bar{x}-\delta, \ldots, \bar{x}+\delta]$

The closest pair is either in the first half of $P_x$, or the second half, or is a "split pair"






recurse - find distance $\delta_1$

recurse - find distance $\delta_2$

$\delta = \min(\delta_1, \delta_2)$.

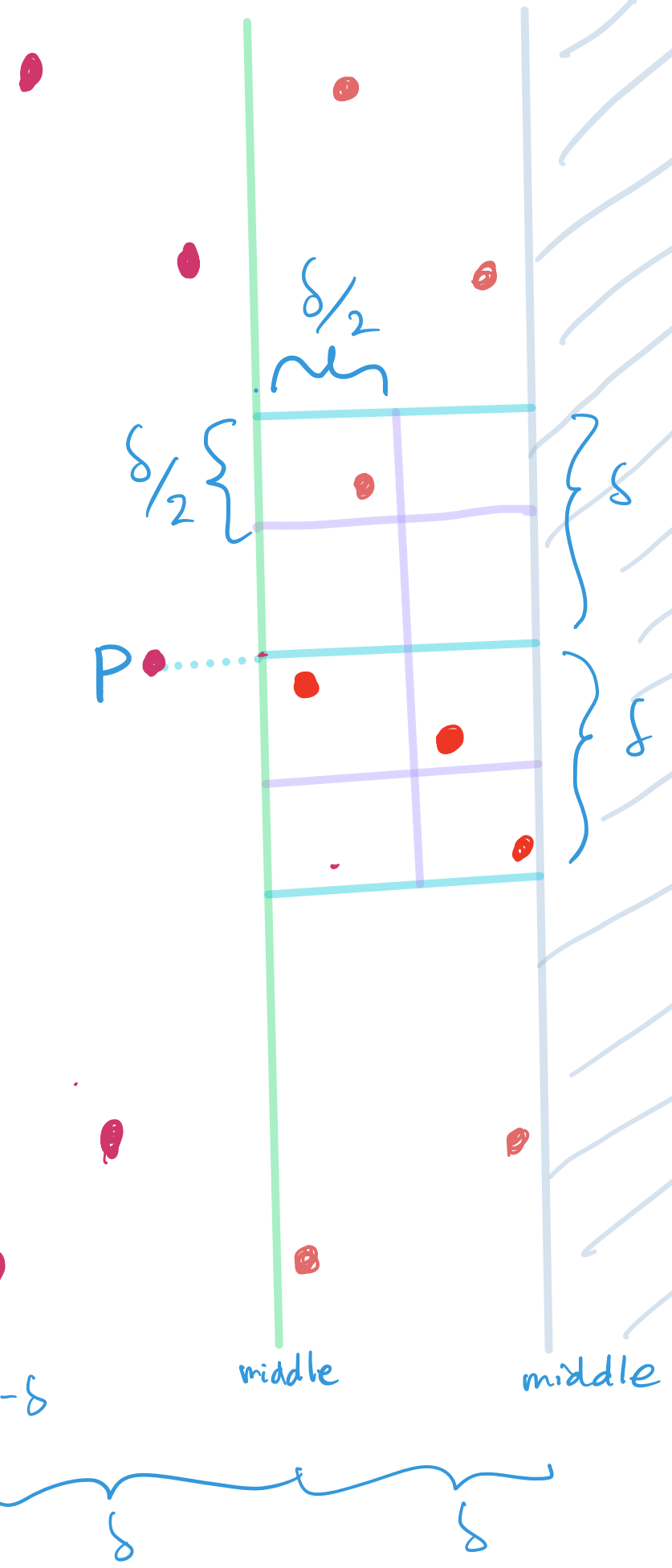
Now find min dist split pair in linear time

- must be within $\delta$ of $\bar{x}$ in $x$-dimension

- for each pt $p$ in that range, check the 8 points that come before or after it in the $y$-dimension (ie in $P_y$).

Why is it sufficient to only look at 8? (in the $y$-direction)?

If ∃ a split-pair partner for p that is is above p and is distance < δ from p is in those boxes

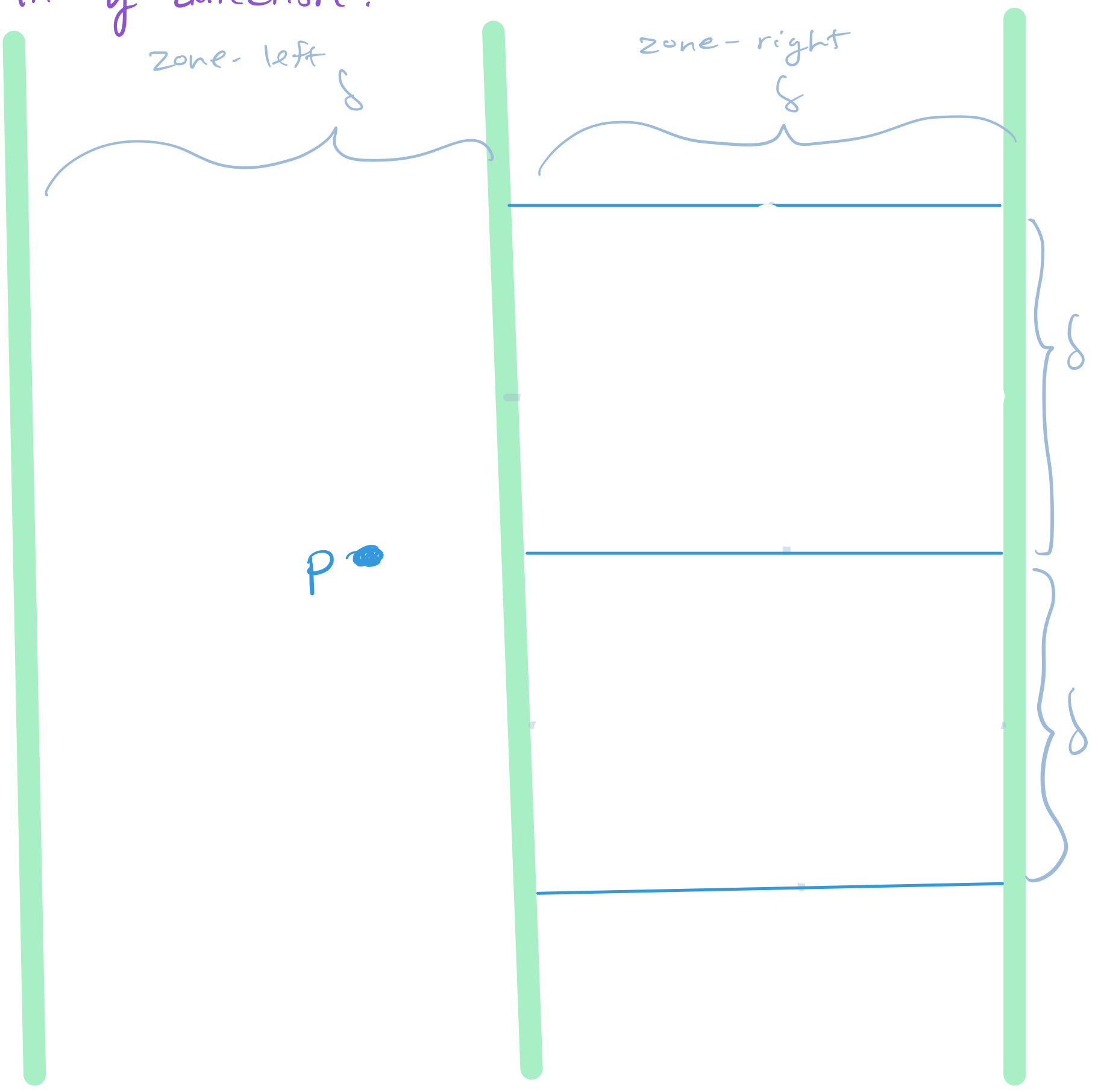



Throw away all these points, Leaving the middle points in order of x-coord and of y-coord

$S_y$ = List of points, sorted by y-coord

Filter $S_y$ of any points not in the $\bar{x} - \delta \ldots \bar{x} + \delta$ zone $O(n)$

For a given point p in (say) the left part of the zone, how many points can be in Right side of zone, and within δ in y-direction?



You need only check the 8 points around $p$ in Zone-filtered $S_y$

This can be done in linear time!

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$n \in \Theta(n')$ ∴ by MT case 2,

$T(n) \in O(n \lg n)$ ▨