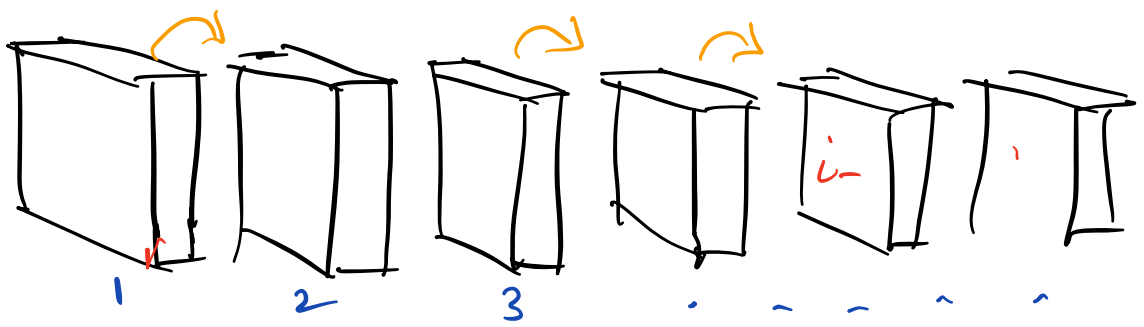
# Induction - Review

Dominoes...




Ind Hyp: $P(i)$ → Prove this: $P(i+1)$

| | |
|---|---|
| "$d_1$ topples" | I. The first domino topples |
| "$d_i$ topples → $d_{i+1}$ topples" | II If the $i^{th}$ domino topples, so does the $i+1^{st}$ |

If we know I and II then...

$\forall n$, we can construct a proof that all dominoes from 1 to n topple:

Rationale

"1. $d_1$ topples ..... I

2. $d_1$ topples → $d_2$ topples ..... II, instantiation

3. $d_2$ topples .....

4. $d_2$ topples → $d_3$ topples ..... II, instantiation

5. $d_3$ topples .....

⋮

$2n-3$. $d_{n-1}$ topples .....

$2n-2$. $d_{n-1}$ topples → $d_n$ topples .....

$2n-2$. $d_n$ topples." .....

Having proven statements

I Base case $P(0)$ or $P(1)$ or $P(\langle$ put base case here $\rangle)$

an II Induction Step $P(i) \rightarrow P(i+1)$ $\forall\, i$ _____ ?

Then you have an ALGORITHM for constructing a proof of $P(n)$ for any $n$, $n \geq$

It must be a statement about integers

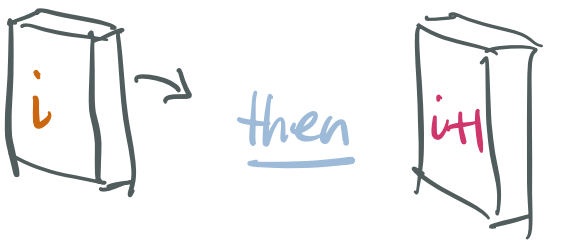
... but statements about graphs can be

Statements about integers...

Statement: All non-empty acyclic connected undirected graphs $T = (V, E)$ have $|E| = |V| - 1$

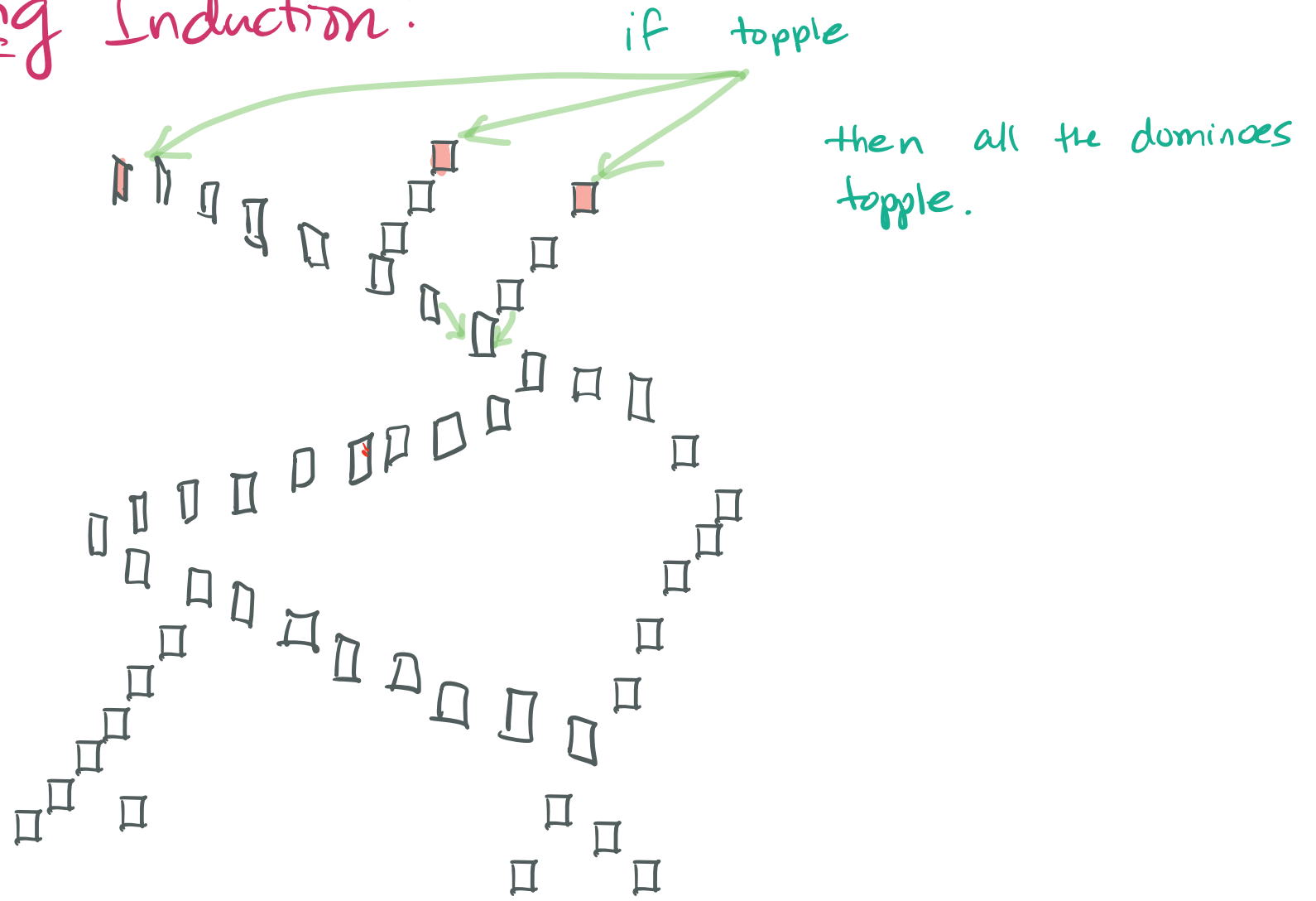
Statement: $\forall\, n$, $n \geq 1$, if $T = (V, E)$ is a non-empty acyclic connected undirected graph on $n$ vertices, then $|E| = |V| - 1$

We would normally use Strong Induction for graphs.

Weak Induction: show <u>if</u> 

<u>then</u>

Strong Induction:




Claim: $\forall n \geq 1$, if $T$ is a tree on $|V|$ vertices, then $T$ has $\underbrace{|V|-1}_{m}$ edges.

Proof:

"Base case: $n=1$, which implies $T = \bullet$

$\rightarrow$ $m=0$.

$\rightarrow$ claim is confirmed for the one & only base case."

Fix value of $n >$ <base case> i.e. $n > 1$ in this case.

(You do not get to assume anything about n's value except $> 1$)

"Suppose n has a value $> 1$"

Induction Hypothesis:

"Claim is true for all ... graphs with $< n$ vertices." OR

"Suppose that, $\forall$ ... graphs $G' = (V', E')$, $|V'| < n$,

$|E'| = |V'| - 1$."

Now you have to prove the claim for the case $|V| = n$. You will have to use the IndHyp.

Let T be any tree on $|V| = n$ vertices.

Then $T \setminus v$ has $n-1$ vertices.

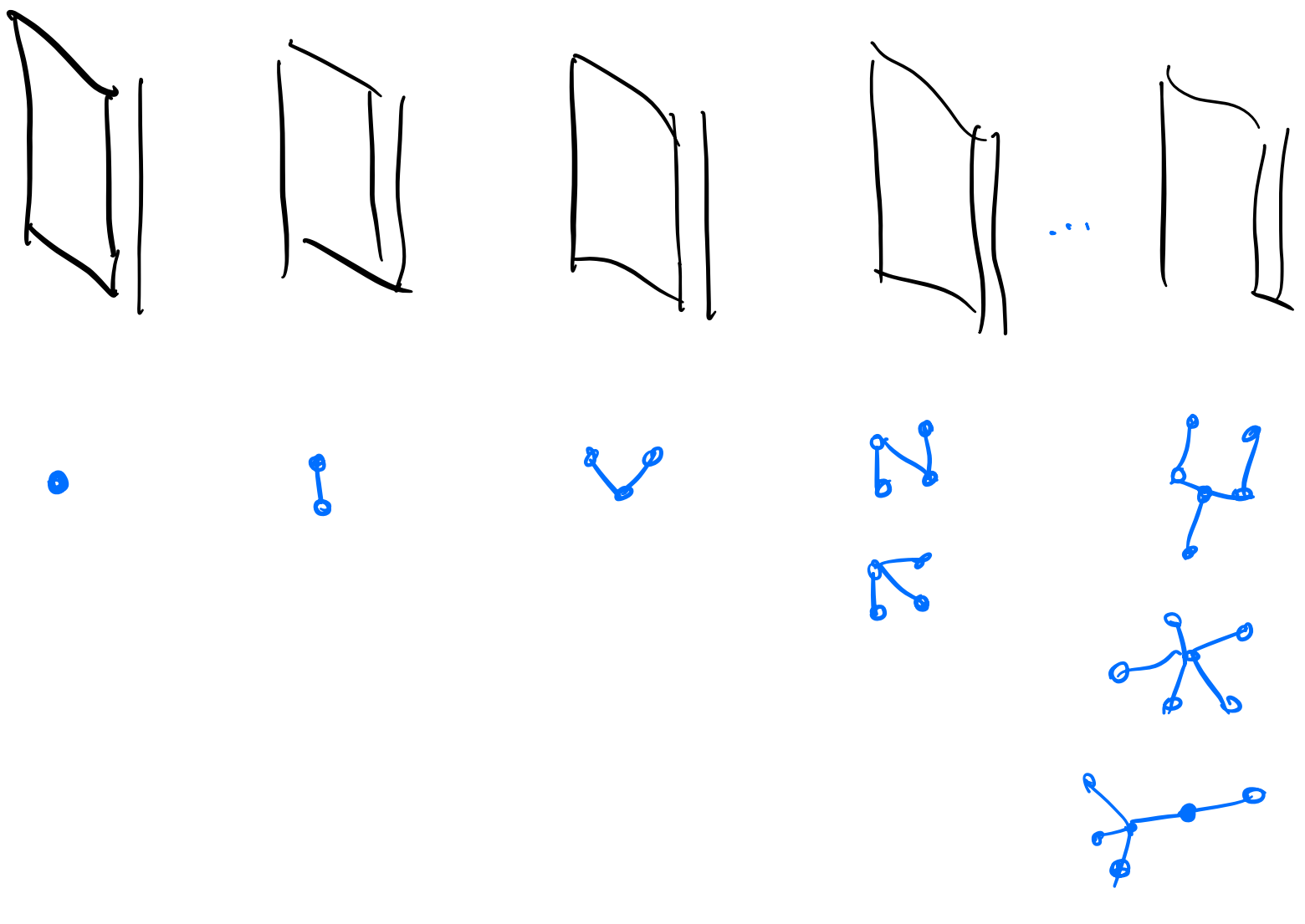
Then the Ind Hyp applies to $T \setminus v$

and $T \setminus v$ has $n-2$ edges. by Ind Hyp

$\Rightarrow$ T has $n-2+1$ edge (adding the one that connected v to rest)

ie T has n vertices and $n-1$ edges.





Claim: $\forall n \geq 1$, every tree with $n$ nodes has $n-1$ edges.

Proof: Strong Induction on number of vertices in the tree.

Base: If $T =$ •, $n = 1$ and num edges $m = 0$.
So claim is true for this case.

Ind step: Let $n'$ be a value $> 1$.

Ind Hyp: $\forall n < n'$, each tree of size $n$ has $n-1$ edges.

Let $T$ be any tree of size $n'$.

Since $n' > 1$, $T$ has at least 2 vertices and at least 1 edge (since $T$ is connected).

Let $(u,v)$ be an edge in $T$.

Then $T \setminus \{(u,v)\}$ is the tree minus the edge, so is a forest of two trees $T_1$ & $T_2$, with number of vertices $n_1$ and $n_2$, respectively.

Note $n_1 + n_2 = n'$, $1 \le n_1 < n'$, $1 \le n_2 < n'$

$\therefore$ $T_1$ has $n_1 - 1$ edges

$T_2$ has $n_2 - 1$ edges.

$\therefore$ number of edges in $T$

$= (n_1 - 1) + (n_2 - 1) + 1 = n' - 1$

($n_1 - 1$: edges in $T_1$; $n_2 - 1$: edges in $T_2$; $1$: $(u,v)$)

