Big O, Big Omega, Theta

Warm-up:

[Recall Thm: $\log_c n < n \quad \forall n \geq 2, \; \forall c \geq 2$]

Prove $3n^2 + 4n\log n + 5 \in O(n^2)$

Proof:

1. $3n^2 \leq 3n^2 \quad \forall n$
2. $4n\log n \leq 4n^2 \quad \forall n \geq 1$
3. $5 \leq n^2 \quad \forall n \geq 3$
4. $3n^2 + 4n\log n + 5 \leq 8n^2$
5. $3n^2 + 4n\log n + 5 \in O(n^2)$, as demoed by $c = \underline{8}$, $n_0 = \underline{3}$

$1 \leq n \quad \forall n \geq 1$

$n \leq n^2 \quad \forall n \geq 1$

Rule 1. Removal of constant factors

$c\,f(n) \in O(f(n)) \quad \forall$ constants $c > 0$

Rule 2. Transitivity of Big O

$f(n) \in O(g(n))$ AND $g(n) \in O(h(n))$

$\implies f(n) \in O(h(n))$

Proof: Suppose $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$.

Then $\exists\, c_1, n_1$ s.t. $f(n) \le c_1 \cdot g(n)\ \forall\, n \ge n_1$

and $\exists\, c_2, n_2$ s.t. $g(n) \le c_2 \cdot h(n)\ \forall\, n \ge n_2$.

$\Rightarrow f(n) \le c_1 \cdot (c_2 \cdot h(n))\ \forall\, n \ge \underline{\max(n_1, n_2)}$

$\Rightarrow f(n) \le (c_1 \cdot c_2) \cdot h(n)\ \forall\, n \ge \underline{\max(n_1, n_2)}$

$\Rightarrow f(n) \in O(h(n))$ as demonstrated by

$c = \underline{c_1 \cdot c_2}$, $\quad n_0 = \underline{\max(n_1, n_2)}$. $\square$

Rule #3. Strange-but-true log domination rule.

$(\log n)^r \in O(n^s)\ \forall\, r,\ \forall\, s > 0.$

Eg $(\log n)^{100000} \in O(n^{0.00001})$

Rule #4. Polynomial Rule

$p(n) \in O(q(n))$, when $p(n)$, $q(n)$ are polynomials in n of degree k and t respectively, $k \le t$

Proof: Let $p(n) = a_k n^k + \dots + a_0 n^0$ $\quad$ a_i s are constants

$q(n) = b_t n^t + \dots + b_0 n^0$. $\quad$ $a_k > 0$, b_t

Observe $\displaystyle \lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$ $\quad$ $\blacksquare$

[Alt defn: $f(n) \in O(g(n))$ if $\displaystyle \lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, for a constant c]

Rule #5. Product Rule

$f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$ $\Longrightarrow f_1(n) \cdot f_2(n) \in O(g_1(n) \cdot g_2(n))$

Proof: pretty easy. Uses, and is analogous to, the fact that

$$\frac{a \leq b \qquad c \leq d}{a \cdot c \leq b \cdot d}$$

Rule #6. Log base is Irrelevant

$$\log_a n \in O(\log_b n) \quad \forall \overset{\text{constant}}{a, b} > 1$$

Proof: $\log_a n = \dfrac{\log_b n}{\log_b a}$ (Theorem)

$\therefore \quad \log_a n = \underbrace{\dfrac{1}{\log_b a}}_{c} \cdot \log_b n$

$\therefore$ by constant factors rule

$$\log_a n = c \cdot \log_b n \in O(\log_b n) \quad \square$$

Rule #7. Reciprocal Rule

$$f(n) \in O(g(n)) \Rightarrow \frac{1}{g(n)} \in O\left(\frac{1}{f(n)}\right)$$

Proof: An exercise for the student. $\square$

Rule #8. Sum Rule

if $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$

$$\Rightarrow f_1(n) + f_2(n) \in O(g(n))$$

Proof: An exercise for the student.

Rule #9 Less-Than Rule

$f(n) \le g(n) \ \forall n \ge n_0$ for some n_0

$\Longrightarrow \ f(n) \in O(g(n))$

Proof: An exercise for the student.

A proof using the rules of Big-O

Claim: $3n^2 \log n - 160 \log^3 n \in O(n^2 \lg n)$

Proof:

		Rationale
1.	$\log n \in O(\lg n)$	Log Base Irrevelant
2.	$3n^2 \in O(n^2)$	CF Rule
3.	$3n^2 \log n \in O(n^2 \lg n)$	1,2, Product Rule
4.	$3n^2 \log n - 160 \log^3 n \in O(3n^2 \log n)$	By Less-Than Rule
5.	$3n^2 \log n - 160 \log^3 n \in O(n^2 \lg n)$	4,3 Transitivity.

$\blacksquare$

When proving a Big O relation using the Rules

- refer to the rule explicitly by name in Rationale
- number the lines of your proof, and use the line numbers in your Rationale

Claim: $4n^2 - 1 \in O\left(\frac{3n^3}{\lg n}\right)$

Proof

1.	$4n^2 - 1 \in O(4n^2)$	Less than rule
2.	$\lg n \in O(n)$	Less-than rule, $n \geq 1$.
3.	$\frac{1}{n} \in O\left(\frac{1}{\lg n}\right)$	2, Recip Rule
4.	$4n^3 \in O(3n^3)$	CF Rule
5.	$\frac{4n^3}{n} \in O\left(\frac{3n^3}{\lg n}\right)$	4, 3 Product Rule.
6.	$4n^2 - 1 \in O\left(\frac{3n^3}{\lg n}\right)$	1, 5 Transitivity

Theorem: If $\lim \frac{f(n)}{g(n)}$ exists, then

$n \to \infty$

$$f(n) \in O(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \le c$$

for constant $c > 0$

if the limit exists

Proof:

$$f(n) \in O(g(n)) \Leftrightarrow \exists\, c, n_0 \text{ s.t. } f(n) \le c \cdot g(n) \ \forall n \ge n_0$$

$$\Leftrightarrow \frac{f(n)}{g(n)} \le c \quad \forall n \ge n_0$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \le c \qquad \blacksquare$$

Def$^{\underline{n}}$ $\Omega(g(n))$

$f(n) \in \Omega(g(n))$ iff $\exists$ positive c, n_0 such that

$$0 \le c \cdot g(n) \le f(n) \quad \forall n \ge n_0$$

let's just consider pos^{ve}-valued functions

Claim: $f(n) \in \Omega(g(n)) \Longleftrightarrow g(n) \in O(f(n))$

Proof:

$f(n) \in \Omega(g(n)) \iff \exists\ c_1, n_1$ s.t. $c_1 \cdot g(n) \le f(n)\ \forall n \ge n_1$

$\iff g(n) \le \frac{1}{c_1} \cdot f(n) \quad \forall n \ge n_1$

$\iff g(n) \in O(f(n))$, as demo'ed

by $c = \frac{1}{c_1}$, $n_0 = n_1$ $\square$.

Def$^{\underline{n}}$: $\Theta(g(n))$

If $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$

then we say $f(n) \in \Theta(g(n))$.

Theta

Claim: $3n^2+1 \in \Theta(n^2)$

Proof: 1. $3n^2+1 \in O(n^2)$ Polynomial rule

2. $n^2 \in O(3n^2+1)$ Polynomial rule.

3. $3n^2+1 \in \Theta(n^2)$ 1, 2 Def$^{\underline{n}}$ Θ. $\blacksquare$