Dynamic Programming (DP) Sept 4 '25

See Jeff Erickson's *Algorithms* Ch 3

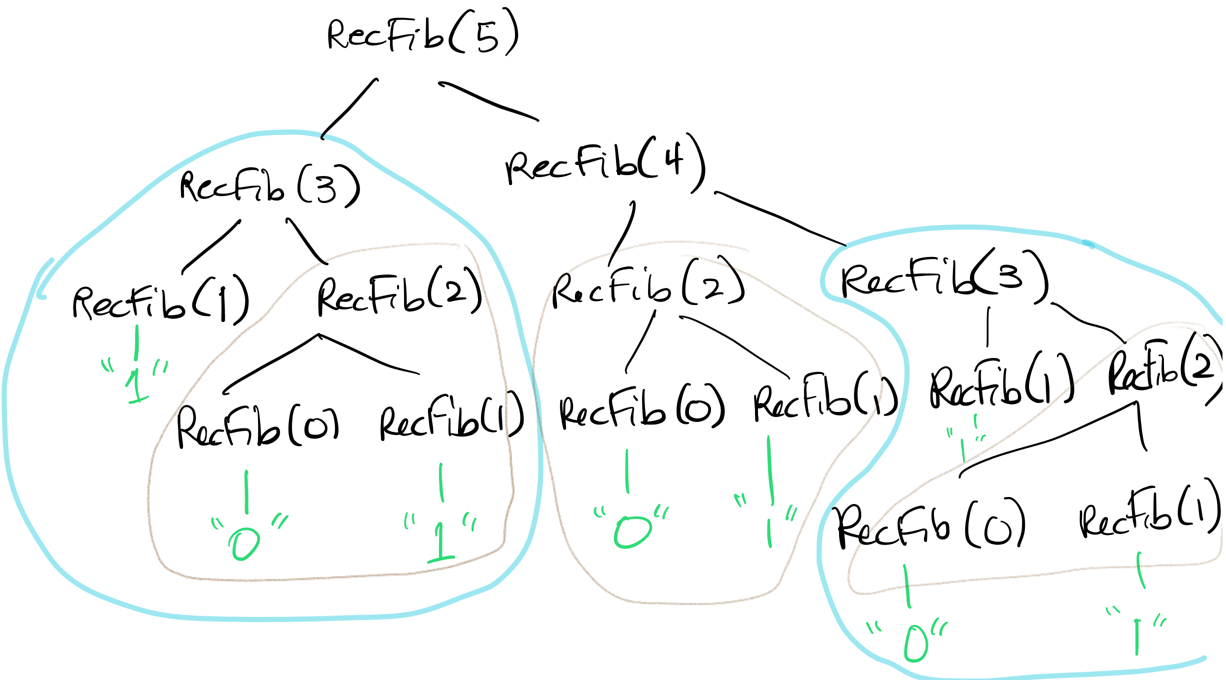
Recall (from 260 intro to DP) that some recursive algorithms may repeatedly compute the same value, to the detriment of the running time:

```
RecFib(n)                // the naive method
    if n = 0
        return 0
    else if n = 1
        return 1
    else
        return RecFib(n-1) + RecFib(n-2)
```

Running time of RecFib is $O(F_n)$

(can show $T(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 1 \\ T(n-1) + T(n-2) + 1 & \text{otherwise,} \end{cases}$

where $T(n)$ = # recursive calls to RecFib on input n.



"Dynamic Programming" = memo-ize data you might use again (don't recompute)

```
global int F[max_n]

MemFib(n)
    if n=0
        return 0
    else if n=1
        return 1

    else
        if F[n] is undefined
            F[n] = MemFib(n-1) + MemFib(n-2)
    return F[n]
```

The above is "top down", a good and necessary direction when you don't know what values of F[i] we will need. But for Fibonacci numbers, we will need all of them ... F[2..n]. So the following also works:

```
IterFib(n)
    F[0] = 0;  F[1] = 1
    for i = 2 to n do
        F[i] = F[i-1] + F[i-2]
    return F[n].
```

IterFib uses $O(n)$ additions and stores $O(n)$ integers.

We have seen one example of DP already... the min Contig Sum problem.
We were able to solve that without $O(n)$ storage... just two integers! Can you do the same with IterFib?

DP is not about filling in tables... it is about smart recursion

↑ this can be implemented as iteration.

How to come up with DP solutions

1. Formulate problem recursively
    - specification – describe coherently
    - solution – clear recursive formula
      – smaller instances of exactly same problem

2. Build solutions from bottom up.
    - identify subproblems
    - choose memoizing structure
    - identify dependencies
    - find good evaluation order ←
    - analyze space & running time.
    - write down algorithm.

Eg. Longest Increasing Subsequence LIS (3.6)

A =

$-\infty$	9	4	18	6	11	13	17	16	19	80	7	21
0	1	2	3	4	5	6	7	8	9	10	11	n

Exhaustive search is exponential time.

all sequences that start with 9

all sequences that start with 9, 18

⋮

all sequences that start with 4

all . . .

If we know stuff about the LIS that starts at 11, that can help us ascertain solutions starting at 9 and at 4.

length of longest increasing subseq that starts at, and includes, A[i]

$$\mathrm{LIS}[i] = \begin{cases} 1 + \max\left(\mathrm{LIS}[j]\right), & \boxed{\begin{array}{l} j > i \\ \text{and } A[i] < A[j] \end{array}} \\ 1 \text{ if } \nexists\, j > i,\ A[i] < A[j] \end{cases}$$

$-\infty$	9	4	18	6	11	13	17	16	19	80	7	21
8	6	7	3	6	5	4	3	3	2	1	2	1

The recursive formula suggests how to compute it.

```
int LongestIncreasingSubsequence(A[1..n])
    // find integer that is length of longest strictly
    // increasing subsequence of A[1..n]
    // (not necessarily contiguous)

    A[0] = -∞
    for i = n downto 0
        LIS[i] = 1          // LIS[i] is length of longest subseq
        for j = i+1 to n    // that starts at (includes) A[i]
            if A[j] > A[i] and LIS[j] ≥ LIS[i]
                LIS[i] = LIS[j] + 1
    return LIS[0] - 1
```

Running time = $O(n^2)$

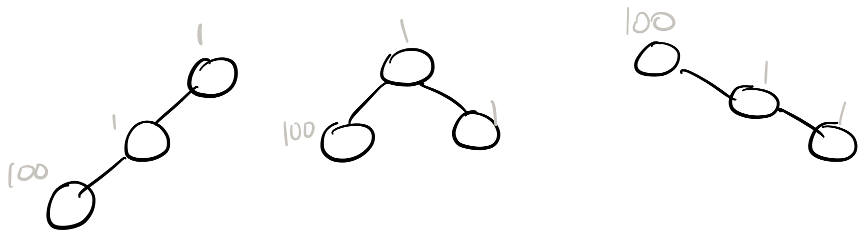
Space = $O(n)$

Optimal BST

index	1	2	3	4	5	6	7	8	9
key:	1	3	8	15	19	22	28	35	44
freq:	60	40	3	19	17	35	19	22	6

We want a binary search tree whose shape is optimal for the frequencies given – ie, makes fewest comparisons in 60 searches for key 1, 40 searches for key 2, etc.

Eg freq = [100, 1, 1]

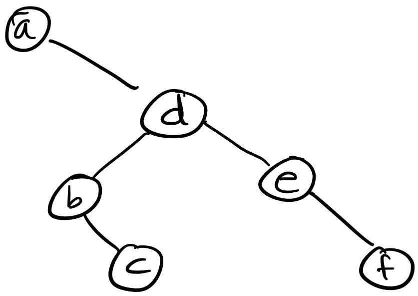


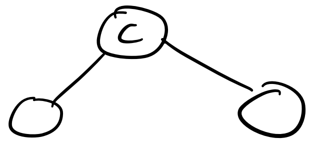
3×100	2×100	1×100
$+ 2 \times 1$	$+ 1 \times 1$	2×1
$+ 1 \times 1$	$+ 2 \times 1$	3×1
302	203	105

tree costs under given frequency assumptions

Given a set of elements from a totally ordered set and the frequencies f that they will be accessed ....

$$A = [\, a \quad b \quad c \quad d \quad e \quad f \,]$$
$$f = [\, 5 \quad 8 \quad 16 \quad 11 \quad 6 \quad 5 \,]$$





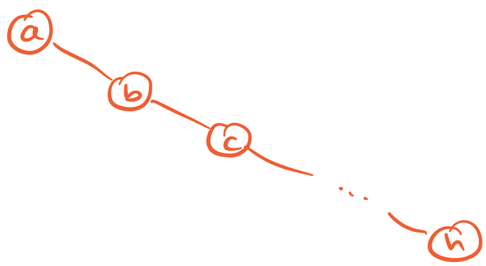
... the <u>cost</u> of a BST that stores those elements is

$$\text{Cost}(T) = \sum_{x \in A} f(x)\,\big(\text{depth}(x) + 1\big)$$

Aside

Why is it not always optimal to just put most frequent at top?

a	b	c	d	e	f	g	h
20	19	18	17	16	15	14	13



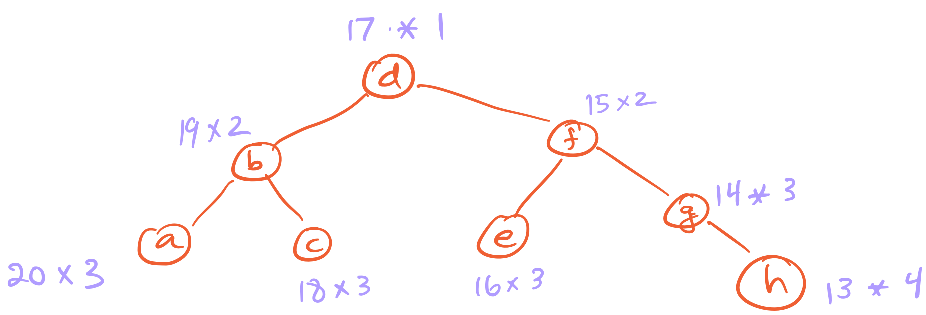
Total comparisons for

20 searches for a: $20 * 1$

19 searches for b: $19 * 2$

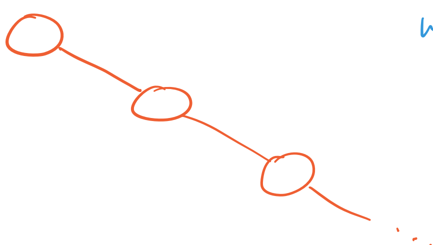
⋮

552



341

What is a frequency profile where a BST like this



would be optimal?

Optimal BST

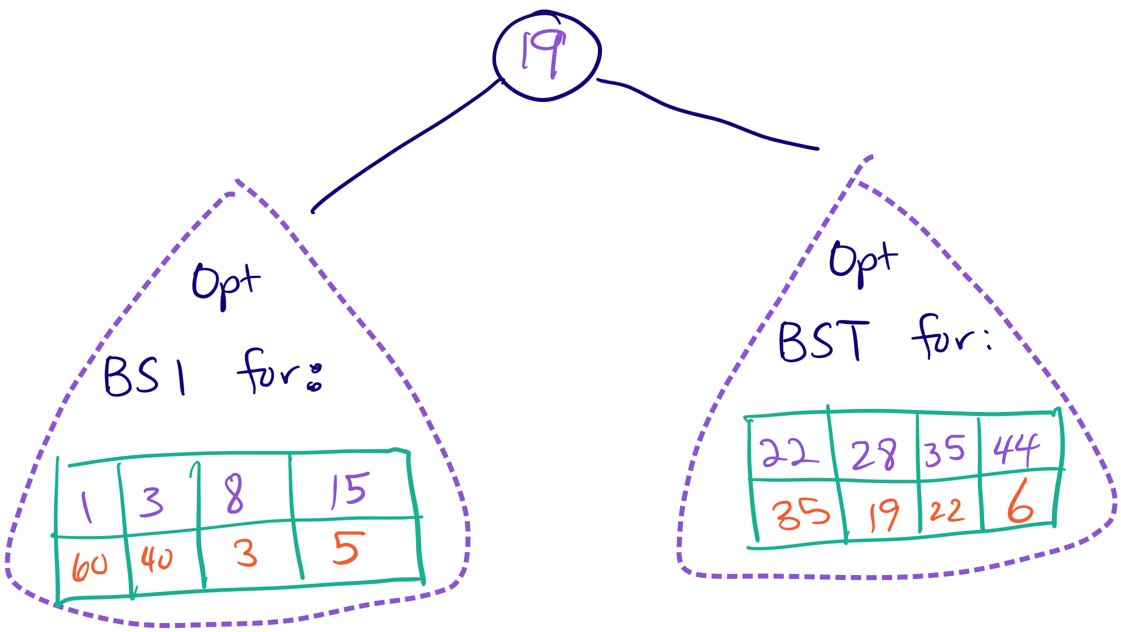
A:

index	1	2	3	4	5	6	7	8	9
key:	1	3	8	15	19	22	28	35	44
freq:	60	40	3	5	17	35	19	22	6

= A

What is the optimal tree cost if A[5] is root?

- useful to know OptCost[1,4] and OptCost[6,9]
- for OptCost[6,9] useful to know OptCost[6,6] and OptCost[8,9], . . .



Optimal BST

A:

index	1	2	3	4	5	6	7	8	9
key:	1	3	8	15	19	22	28	35	44
freq:	60	40	3	5	17	35	19	22	6

= A

What is the optimal tree cost if A[5] is root?

- useful to know OptCost[1,4] and OptCost[6,9]
- for OptCost[6,9] useful to know OptCost[6,6] and OptCost[8,9], . . .

$$\text{OptCost}[i,k] = \begin{cases} 0 & \text{if } i > k, \quad f[i] \text{ if } i = k \\ \displaystyle\sum_{j=i}^{k} f[j] + \min_{i \le r \le k} \begin{cases} \text{OptCost}[i, r-1] \\ + \text{OptCost}[r+1, k] \end{cases} \end{cases}$$

Not

$$\text{OptCost}[i,k] = \begin{cases} 0 & i > k \\ \displaystyle\min_{i \le j \le k} \left(f[j] + \text{OptCost}[i, j-1] + \text{OptCost}[j+1, k] \right) \end{cases}$$

Why?

Optimal BST

Note to self: change to math mode, return later

$$\text{OptCost}[i,k] = \begin{cases} 0 & \text{if } i > k \\ \sum_{j=i}^{k} f[j] + \min\limits_{i \le r \le k} \begin{cases} \text{OptCost}[i, r-1] \\ + \text{OptCost}[r+1, k] \end{cases} \end{cases}$$

Claim: Given any ordered $A[1..n]$ with frequencies $f[1..n]$

the cost of an optimal BST,

$\forall\ i \in \{1, 2, \cdots n\}$, $\forall\ \Delta$ such that $i + \Delta \le n$,

the optimal cost of a BST for the subrange $A[i..i+\Delta]$

is $\text{OptCost}(i, i+\Delta)$ defined above.

Proof: By induction on Δ (i.e. size of range)

Base: $\Delta = 0$.

the "subtrees" have zero cost, so value is

just $f[i]$.

f: 1 8 9 $[11]$ 16 18

4

Induction: $\Delta > 0$

Ind Hyp: $\forall\ \Delta' < \Delta$, claim is true.

How do we compute the function OptCost

$OptCost(1, n-1)$?

First, let's compute $F[i,k] \overset{\text{defn}}{=} \sum_{j=i}^{k} f[j]$ using DP:

$$F[i,k] = \begin{cases} f[i] & \text{if } i = k \\ F[i,k-1] + f(k) & \text{otherwise} \end{cases}$$

recurrence formula

ComputeF ($f[1..n]$)

for $i = 1$ to n

$F[i, i-1] = 0$

for $k = i$ to n

$F[i,k] = F[i, k-1] + f[k]$

F[i]:

0	0	0	0	0	19	36	71	90	112	118

i

$O(n-i)$

f[i] 19 17 35 19 22 6

$$OptCost[i,k] = \begin{cases} 0 & \text{if } i > k \\ F[i,k] + \min\limits_{i \le r \le k} \begin{Bmatrix} OptCost[i, r-1] \\ + OptCost[r+1, k] \end{Bmatrix} \end{cases}$$

Call ComputeF first.

```
ComputeOptCost(i, k)                          // assumes OptCost[i, j<k]
    OptCost[i, k] = ∞                         // and OptCost[j>i, k] have
    for r = i to k                            // been computed already
        tmp = OptCost[i, r-1] + OptCost[r+1, k]   // = smaller ranges.
        if OptCost[i, k] > temp
            OptCost[i, k] = temp
    OptCost[i, k] = OptCost[i, k] + F[i, k]
```

```
OptimalBST(f[1..n])                           // Note: smaller ranges
    ComputeF(f[1..n])                         // are computed first.
    for i = 1 to n+1
        OptCost[i, i-1] = 0
    for d = 0 to n-1
        for i = 1 to n-d
            ComputeOptCost(i, i+d)
    return OptCost[1..n]
```