Fib Heaps (cont'd) 25.11.06

Bounding max degree $D(n)$.

$D(n)$ = max degree in the heap

γ = golden ratio $= \dfrac{(1+\sqrt{5})}{2} = 1.61803\ldots$

size(x) = # nodes in subtree rooted at x.

Lemma 19.1 Let $x \to$ degree $= K$.

Let $y_1 \; y_2 \; y_3 \ldots y_K$ be x's children

in order they were linked to x. Then

WHY?

$y_1 \to$ degree ≥ 0 — Obvious

$y_2 \to$ degree ≥ 0

$y_3 \to$ degree ≥ 1

$y_4 \to$ degree ≥ 2

$\vdots$

$y_i \to$ degree $\geq i-2$

$\vdots$

$y_K \to$ degree $\geq K-2$.

In general, for $i > 0$:
When y_i was linked to x, x already had degree $i-1$ (or more) hence y_i's degree at that time must also have been $i-1$ (or more).
[CONSOLIDATE only links nodes of same degree]. Since that time, y_i can have lost at most one child. $\therefore y_i \to$ degree $\geq i-2$

Fibonacci Numbers

$$F_k = \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k=1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Can prove by induction that:

Lemma 19.2 $\forall\ k \geq 0 \quad F_{k+2} = 1 + \sum_{i=0}^{k} F_i$

Lemma 19.3 $\forall\ k \geq 0 \quad F_{k+2} \geq \gamma^k$

Lemma 19.4 Let x be any node in a Fib Heap

Let $k = x \rightarrow \text{degree}$

Then $\text{size}(x) \geq F_{k+2} \geq \gamma^k$,

where $\gamma = \dfrac{(\sqrt{5}+1)}{2}$

Proof: Let S_k denote min possible size of any node of degree k in any Fib Heap

Claim: $S_k \geq F_{k+2} \quad \forall\ k \geq 0$. ← Hyp thm for all deg

Proof: By Induction on k.

Basis: $S_0 = 1$
$S_1 = 1$ } Trivial

Note that $S_0 < S_1 < S_2 \ldots < S_k$

Let z be a node that realizes the min for degree k.

$z \to degree = k$ and

$size(z) = S_k$

Let $y_1 \quad y_2 \quad y_3 \; \ldots \; y_k$ be z's children

in order they were linked to z.

$$size(z) = S_k$$

$$\geq 2 + \sum_{i=2}^{k} S_{y \to degree}$$

z, and y_1

$$\geq 2 + \sum_{i=2}^{k} S_{i-2} \text{, from Lemma 19.1}$$

By Ind Hyp, this is $\geq F_i$

$$\geq 1 + \sum_{i=0}^{k} F_i$$

took 1 from here

added terms "0 + 1 +" to start of this sequence

$$= F_{k+2} \qquad \text{by Lemma 19.2}$$

$$\geq \gamma^{k} \qquad \text{by Lemma 19.3.} \qquad \blacksquare$$

Corollary 19.5 The max degree $D(n)$ of any node in an n-node Fib Heap is $O(\lg n)$

Proof: $\forall$ nodes x

$$n \geq \text{size}(x) \geq \gamma^{D(n)}$$

Take $\log_\phi$ of all terms.

$$\therefore D(n) \leq \log_\gamma n$$

$$\therefore D(n) \in O(\lg n). \qquad \blacksquare$$

	Fib Heap	Binary Heap	Binomial Heap	Leftist Heap
Insert				
Extract Min				
Merge				
Decrease Key				