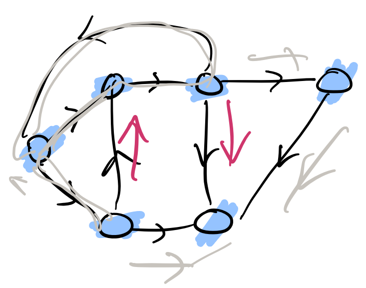
Network Flow: Disj. Path Covers 11/06

Jeff Erickson's "Algorithms", Ch. 11.5

A "path cover" of a digraph G is a collection of directed path in G such that each vertex lies on ≥ 1 path. It is a disjoint path cover if each vertex lies on $!1$ ("exactly one") path.

$\forall$ digraph has such a cover: let the path lengths = 0.

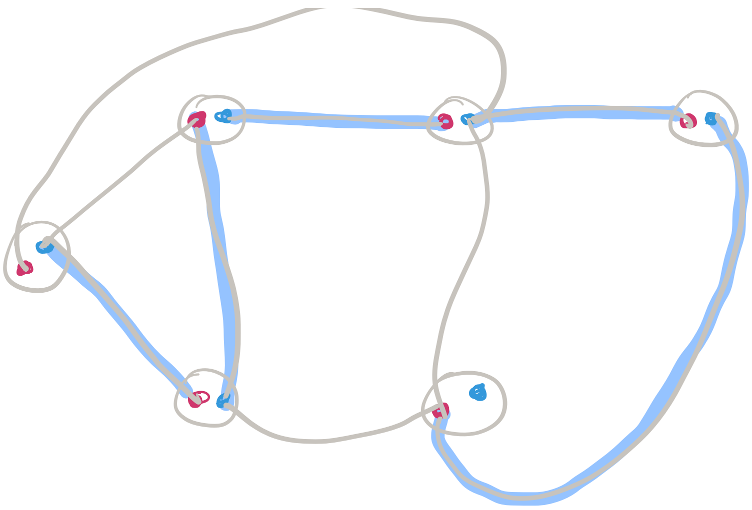


But we want fewest paths that cover V and are disjoint

NP-c in general

What if we restrict to DAGs? (directed acyclic graphs)

To solve using NF



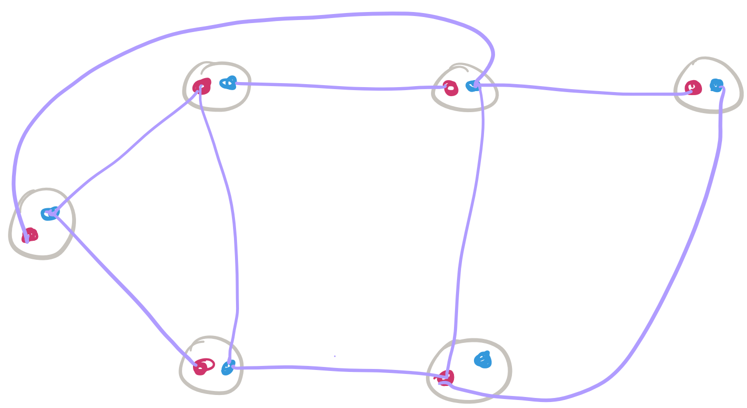
Create a new graph

- each v

becomes v v

- if $u \to v$

then $u \to v$



G'

Create a new graph

- each v becomes v v

- if $u \to v$ then $u \to v$

G' has a matching of size K $\iff$ G has disjoint path cover of size $n-K$

Proof:

($\Rightarrow$) ① G' is bipartite, since the vertices can be partitioned into red and blue partitions, and all edges are between a red + a blue vertex.

② Let K = # of unmatched red vertices

Then, since # red vertizes = # blue vertizes, K also = # of unmatched blue vertices.

③ $K > 0$ since otherwise G is cyclic (you will prove this on assignment 3.)

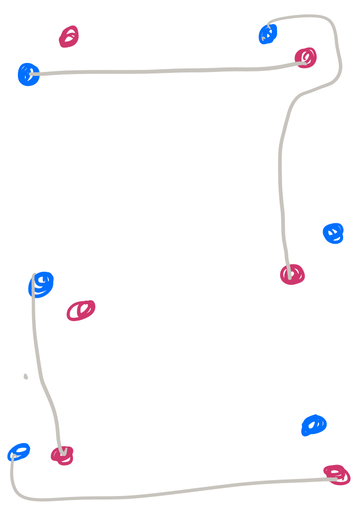
④ We will construct a path cover of size $n-K$ in G, guided by the matching in G', as follows:

(and for each $v \in G$, we have v $v \in G'$

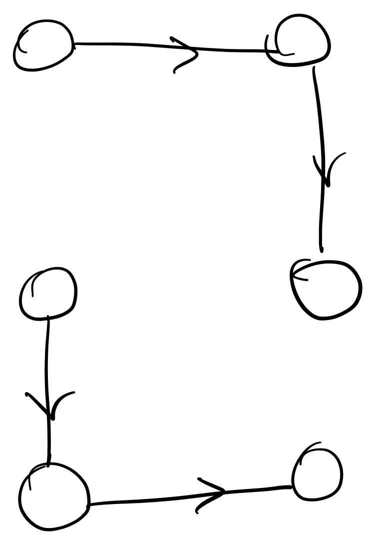
a) $\forall$ unmatched $v \in G$, take the following path: $v \to u_1 \to u_2 \to \dots \to u_z$

where: $v \to u_1$ is a matching edge.

$u_1 \to u_2$ is a matching edge, etc.



matching of size 4 in G'

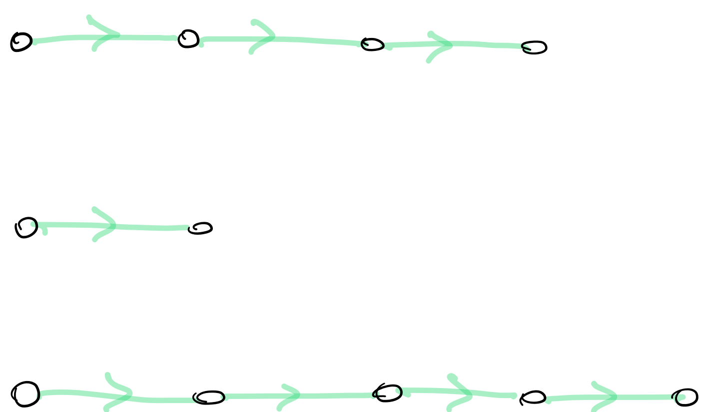


path cover of size $6-4$ in G

The result is a path cover of G of size = # of "path starts" ie $= \underbrace{n-K}$

because K blue vertices NOT in the matching are "path starts".

($\Leftarrow$) Suppose the DAG G can be covered by K disjoint paths



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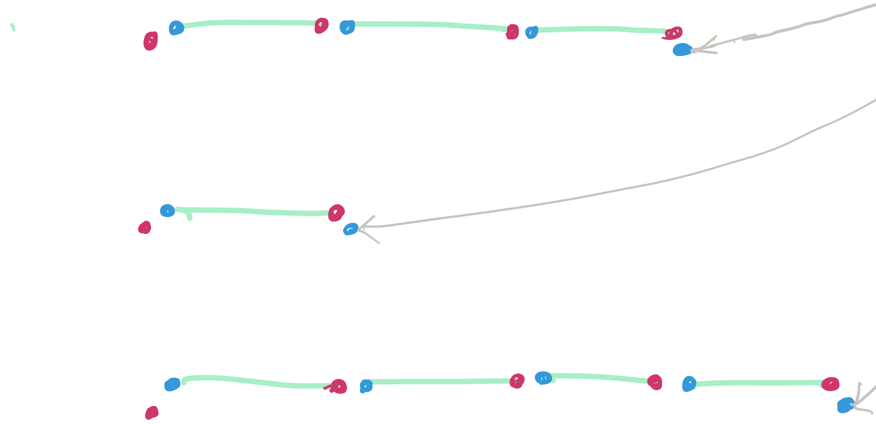

K blue verts
unmatched
and K
red verts
unmatched.

Then each of the path edges from G will
match a ●-vertex to a ● vertex

The size will be $\dfrac{2n - 2K}{2}$ or $n-K$

total
verts

each
edge
matches 2
verts

unmatched.
verts