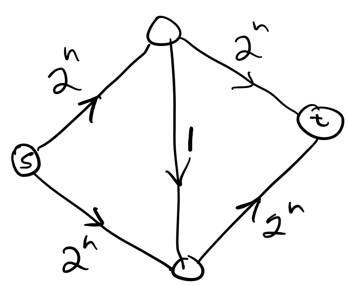
Network Flows (cont'd) 10.30

Network Flow via flow augmenting paths can be exponential time if we are not discriminating in selecting our flow augmenting path, as the following example illustrates:

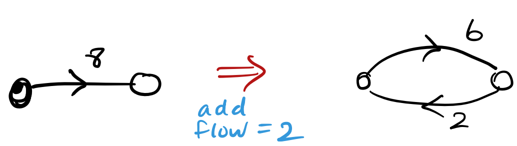


$n=4$ $m=5$
vertices edges

input size:
"bad choices" running time:

Shortest Augmenting Paths

- Choose the flow-augmenting path with the smallest number of edges
  [use BFS on residual graph in a smart way
  i.e. either update the graph to show backwards edges or allow backwards traversal of flow-positive edges]



Theorem: Shortest-Augmenting Paths finds max flow in $O(VE^2)$ time

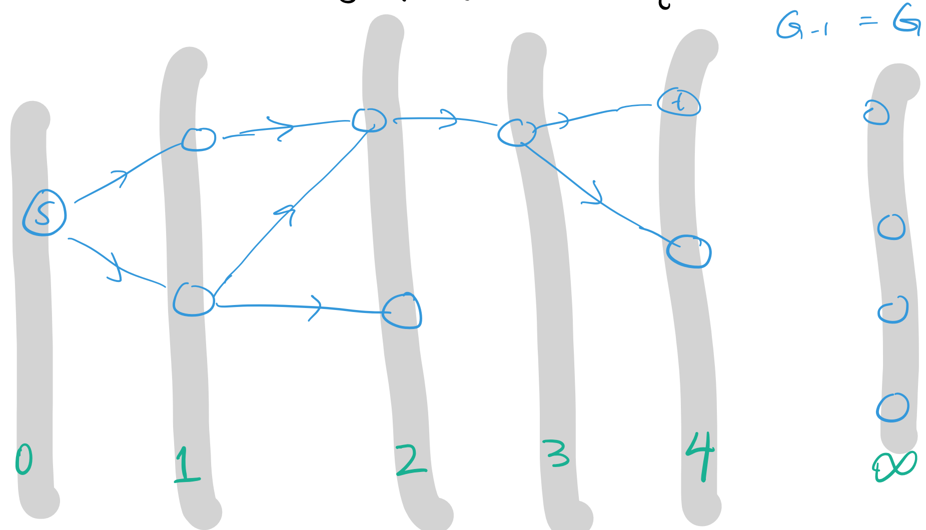
Proof:

Let f_i = flow after i iterations
(i.e. after the i^{th} f.a. path has been added to f)

G_i = residual graph after i iterations

$\text{level}_i(v)$ = unweighted shortest-path distance from s to v in G_i

$G_0 = G$ (no flow)
$G_{-1} = G$ (no flow)



Lemma 10.2 (Erickson's text) $\text{level}_i(v) \geq \text{level}_{i-1}(v)$
$\forall v \in V, \ \forall i > 0$

Proof of Lemma 10.2

Fix a vertex v.

We will show $\text{level}_i(v) \geq \text{level}_{i-1}(v) \quad \forall i = 0, 1, 2, \ldots$

by Induction on level (i.e. dist from s, in edges)

Basis: The only node at level 0 is s, and it is always at level 0.

Fix a level $\ell > 0$

Our claim is that $\forall$ nodes at level ℓ (at any i) then in the previous iteration they were at level ℓ or less

Now consider a vertex v that is at level ℓ at iteration i.

Note: $v \neq s$

Also, if $\text{level}_i(v) = \infty$, claim is true

So we only need consider the case where in G_i $\exists$ a shortest path

$$s \to \cdots \to u \to v$$

u is at a level $< \ell$

$\therefore \text{level}_i(v) = \text{level}_i(u) + 1$

and $\text{level}_i(u) \geq \text{level}_{i-1}(u)$

Need to show $\text{level}_{i-1}(u) \geq \text{level}_{i-1}(v) - 1$

$\text{level}_{i-1}(u) + 1 \geq \text{level}_{i-1}(v)$

- if $u \to v$ is an edge in G_{i-1}

  then $\text{level}_{i-1}(v) \leq \text{level}_{i-1}(u) + 1$ – done

- if $u \to v$ is not an edge in G_{i-1}

  $\Rightarrow$ $v \to u$ is an edge in i^{th} aug. path

  By defn, that is on shortest path

  from $s \to \dots \to v \to u \to \dots \to t$

  in G_{i-1}

  $\therefore \text{level}_{i-1}(v) = \text{level}_{i-1}(u) - 1$

  $\leq \text{level}_{i-1}(u) + 1$

$\blacksquare$ Lemma

Lemma 10.3 During the execution of "Edmonds Karp" shortest flow-augmenting path alg, each edge $u \to v$ disappears from residual graph at most $\frac{n}{2}$ times $(n = |V|)$

Proof: Suppose $u \to v$ is in G_i and G_{j+1}

but not in $G_{i+1}, \dots, G_j$

$\Rightarrow$ $u \to v \in$ the i^{th} aug. path, $\text{level}_i(v) = \text{level}_i(u)+1$

and $v \to u \in$ the j^{th} aug path, $\text{level}_j(v) = \text{level}_j(u) - 1$

$\therefore$ by Lemma 10.2, $\text{level}_j(u) = \text{level}_j(v)+1 \geq \text{level}_i(v)+1$

$= \text{level}_i(u)+2.$

$\therefore$ in between disappearerce and reappearance of $u \to v$, distance from s to u has increased by ≥ 2.

All levels are $< n$ (or infinite),

So can disappear/reappear $\leq \frac{n}{2}$ times. $\blacksquare$ lemma

Now we can prove our theorem.

In run of Shortest Augmenting Paths

- $\exists$ at most $m \cdot \frac{n}{2}$ edge disappearances

– ≥ 1 edge disappears at each iteration (WHY?)

$\therefore$ there are $\leq m \cdot \frac{n}{2}$ iterations

each iteration runs in $O(m)$

$\therefore$ running time is $O(n\,m^2)$

$\boxed{}$ Thm.

Further progress

Very active area of research.

Technique	Direct	with Dynamic Trees
Blocking Flow	$O(n^2 m)$	$O(n\,m \lg n)$
Network Simplex	$O(n^2 m)$	$O(n\,m \lg n)$
Push-relabel	$O(n^2 m)$	—
Push-Relabel (FIFO)	$O(n^3)$	$O(n\,m \lg(\frac{n^2}{m}))$

Push-relabel highest label	$O(n^2 \sqrt{m})$	—
Push-Relabel-Add Games	—	$O(n m \log_{m/n \log n} m)$
Pseudoflow	$O(n^2 m)$	$O(n m \log n)$
Pseudoflow (Highest label)	$O(n^3)$	$O(n m \log(n^2/m))$
Incremental BFS	$O(n^2 m)$	$O(n m \log(n^2/m))$
Compact Networks	—	$O(n m)$ Orlin 2012
Shortest Aug-Path	$O(n m^2)$	