Fib Heaps (cont'd) 10.16

Bounding max degree $D(n)$.

$D(n)$ = max degree in the heap

γ = golden ratio $= \frac{(1+\sqrt{5})}{2} = 1.61803...$

$size(x)$ = # nodes in subtree rooted at x.

Lemma 19.1 Let $x \to degree = K$.

Let $y_1 \quad y_2 \quad y_3 \; \cdots \; y_K$ be x's children

in order they were linked to x. Then

$y_1 \to degree \geq 0$

$y_2 \to degree \geq 0$

$y_3 \to degree \geq 1$

$y_4 \to degree \geq 2$

$\vdots$

$y_i \to degree \geq i-2$

$\vdots$

$y_K \to degree \geq K-2$.

WHY?

Obvious

In general, for $i > 0$:

When y_i was linked to x, x already had degree $i-1$ (or more) hence y_i's degree at that time must also have been $i-1$ (or more). [CONSOLIDATE only links nodes of same degree]. Since that time, y_i can have lost at most one child. $\therefore y_i \to degree \geq i-2$

Fibonacci Numbers

$$F_k = \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k=1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Can prove by induction that:

Lemma 19.2 $\forall\ k \geq 0 \quad F_{k+2} = 1 + \sum_{i=0}^{k} F_i$

Lemma 19.3 $\forall\ k \geq 0 \quad F_{k+2} \geq \phi^k$

Lemma 19.4 Let x be any node in a Fib Heap

Let $k = x \to \text{degree}$

Then $\text{size}(x) \geq F_{k+2} \geq \phi^k$,

where $\phi = \frac{(\sqrt{5}+1)}{2}$

Proof: Let $s(k)$ denote min possible size of any node of degree k in any Fib Heap

Claim: $s(k) \geq F_{k+2} \quad \forall\ k \geq 2$.

Proof: By Induction on k.

Basis: $S_0 =$ } Trivial
$S_1 =$

Note that $S_0 < S_1 < S_2 \ldots < S_k$

Let z be a node that realizes the min for degree k.

$z \to$ degree $= k$ and
size$(z) = S_k$

Let $y_1 \quad y_2 \quad y_3 \quad \ldots \quad y_k$ be z's children

in order they were linked to z.

$$\text{size}(z) = S_k$$
$$\geq 2 + \sum_{i=2}^{k} S_{y \to \text{degree}}$$
$$\geq 2 + \sum_{i=2}^{k} S_{i-2}\text{, from Lemma 19.1}$$

By Ind Hyp, this is $\geq F_i$

$$\geq 1 + \sum_{i=0}^{k} F_i$$

took 1 from here

added terms "0 + 1 +" to start of this sequence

$= F_{k+2}$ by Lemma 19.2

$\geq \phi^k$ by Lemma 19.3. $\blacksquare$

Corollary 19.5. The max degree $D(n)$ of any node in an n-node Fib Heap is $O(\lg n)$

Proof: $\forall$ nodes x

$$n \geq size(x) \geq \phi^{D(n)}$$

$$\therefore D(n) \leq \log_\phi n$$

$$\therefore D(n) \in O(\lg n). \blacksquare$$