Fibonacci Heaps cont'd 10.11.

Fibonacci Heaps use "lazy" implementation of Insert, min, and union – if you never have an Extract Min, then its easy to just keep every node in the root list and a pointer to the Min.

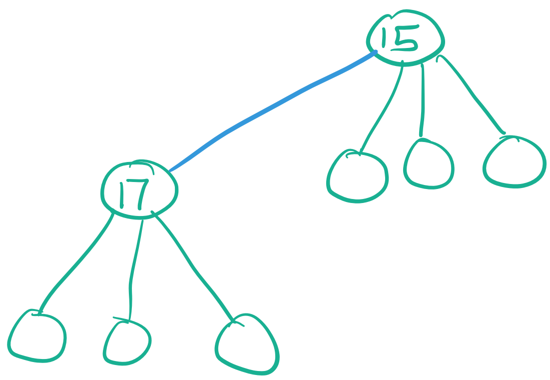
However, this means that the root list is largely unstructured – there is no way to find the next Min after an Extract Min except to conduct a linear search of the root list.

General Amortization Strategy

When you have to do a large amount of work (like $O(r)$, where r is # nodes in the root list) also do $O(r^*)$ amount of "clean up" – ie reduce the potential for future work.

What we want CONSOLIDATE to accomplish?

- find the min root in rootlist
- merge trees in root list so there are fewer to search in future

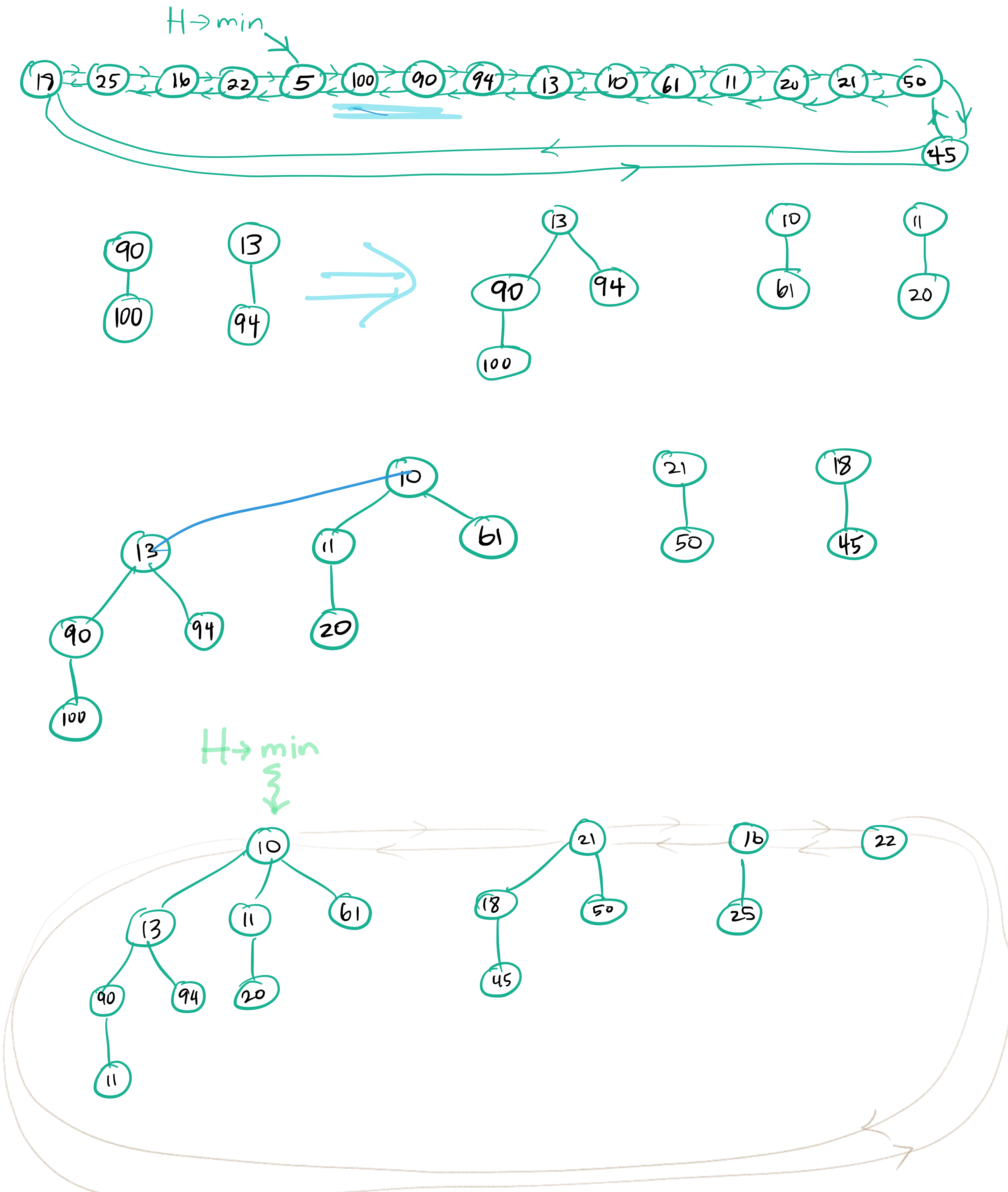


Find 2 trees of same degree d

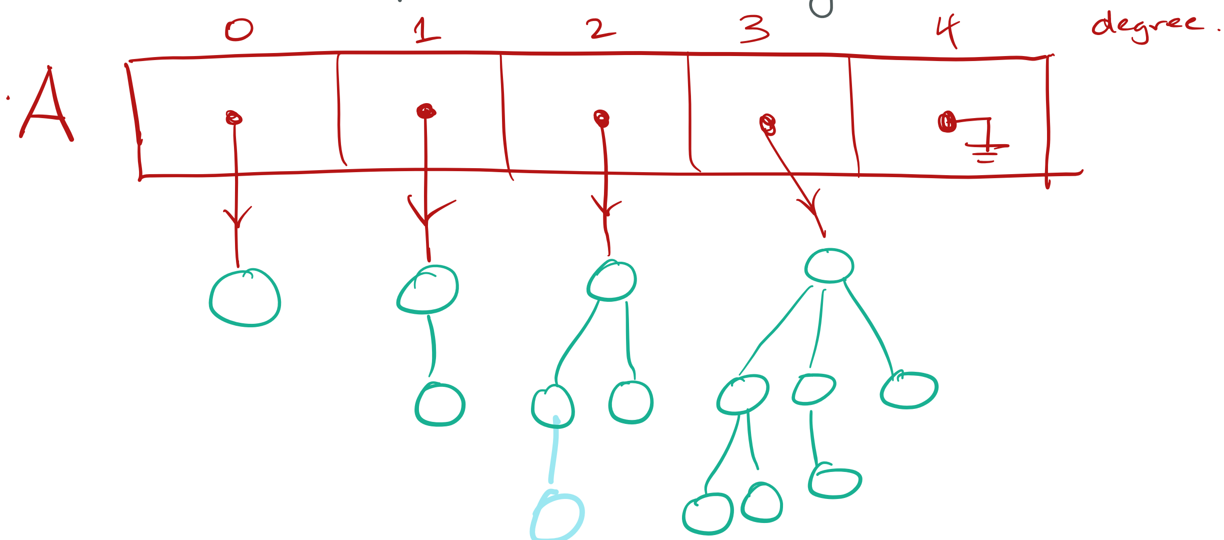
Make one be child of other, creating a $d+1$ degree tree

At the end, we want there to be 0 or 1 tree of each degree

How we want it to work, if we do l inserts and then an Extract Min :



How do we accomplish this efficiently in code?



$A[0 \,..\, D(n)]$ is an auxiliary array that is created during the CONSOLIDATE op'n

$D(n)$ is the maximum degree of any root in the heap of n nodes

CONSOLIDATE (H)

```
new A[0.. D(H,n)] of pointers to trees
for i = 0 to D(H,n)   A[i] = NULL
for each w in H→rootlist          ← t(H)
    x = w;      d = x → degree
    while A[d] ≠ NULL
        y = A[d]
        if  x→key > y→key
            exchange x with y
            // now x should be merged tree's root
        FibHeapLink ( H, y, x )
        A[d] = NULL
        d = d+1
    A[d] = x
H→min = NULL
for i = 0 to D(n)                 O(D(n))
    if A[i] ≠ NULL
        if H→min == NULL
            create rootlist just containing A[i]
            H→min = A[i]
```

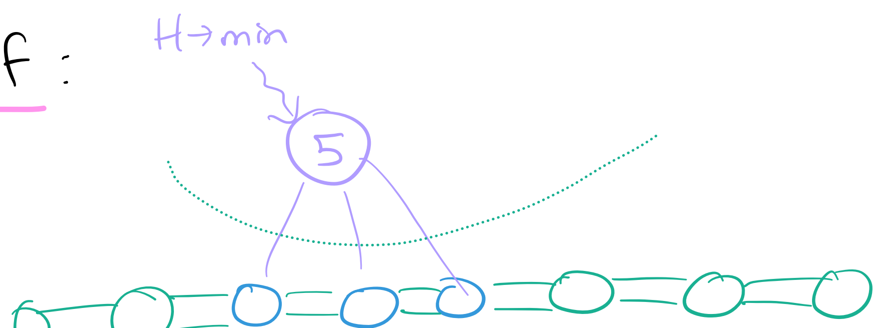
else
    insert A[i] into H's rootlist
    if A[i]→key < H→min→key
        H→min = A[i]

FibHeapLink(H, y, x)

    remove y from root list of H
    make y a child of x
    x→degree++
    y→mark = FALSE   //"mark" only if node has lost
    // 2 children since it got its current parent

Claim: Amortized cost of ExtractMin is $O(D(n))$

Proof:



Actual cost to add H→min's children to root list is $O(D(n))$

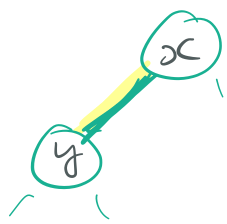
for each w in rootlist
    ....

Let's count up the work in this part - Aggregate Analysis.

Size of rootlist at this point is

$$\leq \underbrace{t(H) - 1}_{\text{roots except extracted min}} + \underbrace{D(n)}_{\text{new roots that were children of min}}$$

- The while loop is $O(1)$
- each time it gets executed, the number of trees in root list is diminished by 1



$\therefore$ total number of executions of <u>while</u> loop is $\leq t(H) - 1 + D(n)$ and each execution is $O(1)$

$\therefore$ total work done in Extract Min is $O(D(n) + t(H))$.

Also, $\Delta\Phi = \underbrace{D(n)+1 + 2m(H)}_{\Phi \text{ after}} - \underbrace{(t(H) + 2m(H))}_{\Phi \text{ before}} = D(n)+1-t(H)$

$\therefore$ work + $\Delta\Phi \in O(D(n) + t(H))$

$$+ O(D(n) + 1 - t(H))$$

$$\in O(D(n))$$

(we scale up the units of potential to dominate the constant hidden in $O(t(H))$)

Claim: $D(n) \in O(\lg n)$

Proof: later

Corollary: Extract Min has amortized running time $O(\lg n)$

Decrease Key and Delete

Decrease key (H, x, k)

if $k \geq x \to key$ error("new key not less")

$x \to key = k$

$y = x \to p$

if $y \neq$ NULL and $x \to key < y \to key$

CUT (H, x, y)

CASCADING CUT (H, y)

if $x \to key < H \to min \to key$

$H \to min = x$

CUT (H, x, y)

remove x from y's child list; $y \to degree$ --

add x to H's rootlist

$x \to p =$ NULL

$x \to mark =$ FALSE

$\Theta(1)$

CASCADING CUT (H, y)

$z = y \to p$

if $z \neq$ NULL

if $y \to mark ==$ FALSE

$y \to mark = TRUE$

else

CUT(H, y, z)

CASCADINGCUT(H, z)

How DecreaseKey works.

- constant amount of work, $\Delta\Phi = 0$ UNLESS it leads to violation of Heap Order.

If $x \to key < x \to p \to key$ then

- "cut" x (from $x \to p$) (put x in rootlist)
- if $x \to p$ has already had a child cut then
  - "cut" $x \to p$ from $x \to p \to p$
  - if $x \to p \to p$ has already had a child cut
    - "cut" $x \to p \to p \to p$

⋮

We use "mark" to tell us whether a node has already had a child cut. — only allowed 1 child cut since it got its current parent

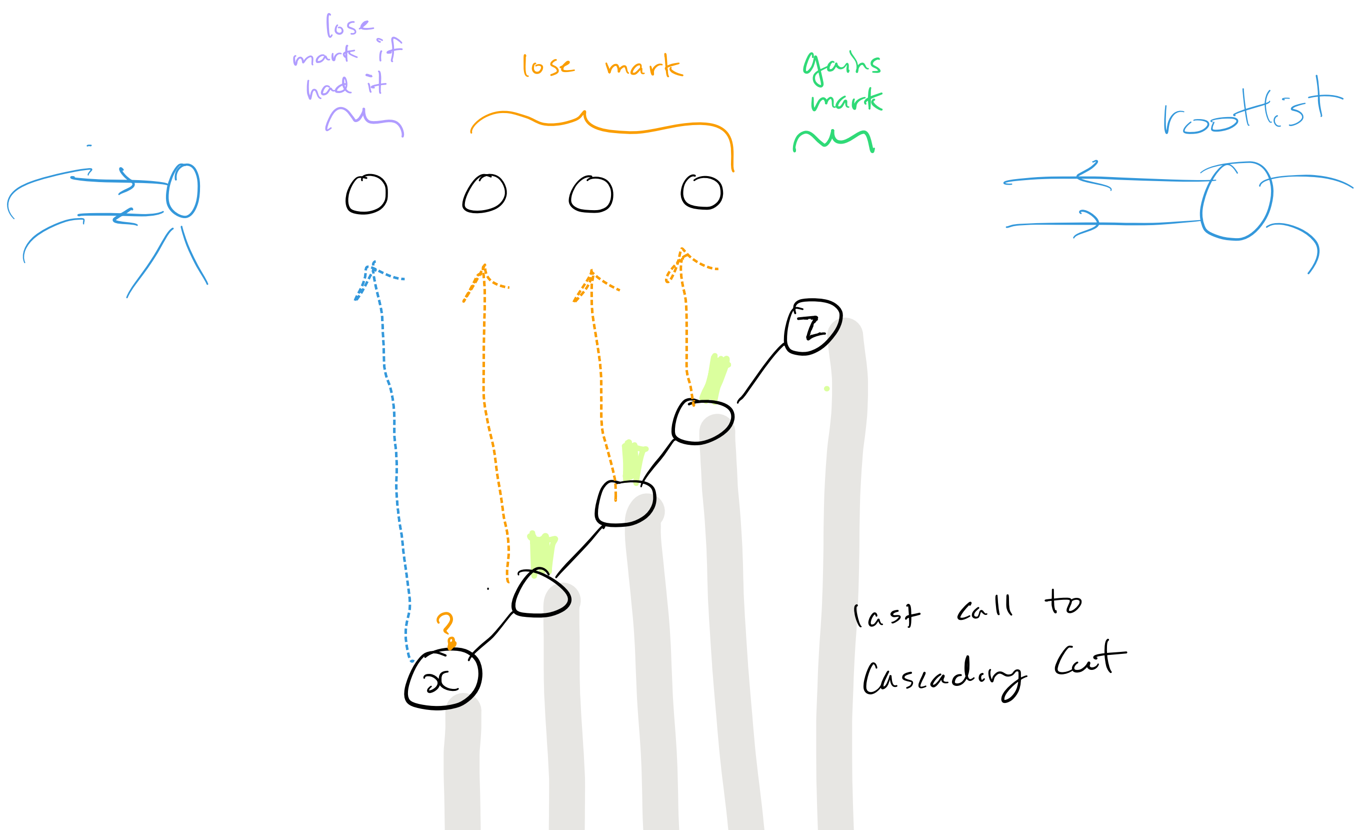
Claim: Amortized cost of Decrease Key is $\Theta(1)$

Proof: Recall $\Phi = t(H) + 2m$

Decrease Key is $\Theta(1)$ if no cascading cuts
(i.e. either no cuts or just x is cut)
- reduces m by 0 or 1

Suppose Decrease Key prompts c cascading cuts
(of $x \to p$, $x \to p \to p$, etc)

Of the c calls to cascading cut,
- the child's "mark" was true
and gets set to false



$\Delta\phi$						
	t:	+1	+1	+1	+1	0
	2m:	(0 or -2)	-2	-2	-2	0 or 2 ?
	work:	+1	+1	+1	+1	+1

$$2 + 0 + 0 + 0 + 3$$

$\Delta\phi$ + work done $\leq$

$\therefore$ the DecreaseKey operation costs

$$\text{work} + \Delta\phi = \Theta(1) \quad \text{work in DecreaseKey } \underline{\text{not}} \text{ in CUT + CASCADCUT}$$

$$+ \Theta(1) + 5 \quad \text{work in CUT and CASCADING CUT, aggregating the recursive calls.}$$

$$= \Theta(1) \text{ amortized time.}$$

CLRS

Delete (H, x)

DecreaseKey $(H, x, -\infty)$

ExtractMin (H)

DecreaseKey is $\Theta(1)$ amortized

ExtractMin is $\Theta(D(n))$ amortized

$\therefore$ Delete is $\Theta(D(n))$ amortized.