Computing Science 320 Quiz 1 SOLUTIONS – April 10, 2025

1. (a) (6 marks) The following Boolean Expression Φ is in Conjunctive Normal Form, and could be the input for SAT decider. Convert it, using the algorithm we studied, into a Boolean Expression Φ' appropriate for 3SAT.

$$\Phi = (s \vee t \vee u \vee v \vee w \vee x \vee y \vee z)$$

ANSWER:

$$\Phi' = (s \vee t \vee A) \wedge (\overline{A} \vee u \vee B) \wedge (\overline{B} \vee v \vee C) \wedge (\overline{C} \vee w \vee D) \wedge (\overline{D} \vee x \vee E) \wedge (\overline{E} \vee y \vee z)$$

(b) (2 marks) Suppose w is true and s, t, u, v, x, y, z are false. Is the expression in your answer above satisfiable under these assumptions? If so, give an assignment of truth values to the other variables in the expression so as to make Φ' true.

ANSWER: Φ' is satisfied by extending the given assignment as follows:

$$A = B = C = TRUE; \quad D = E = FALSE$$

2. (8 marks) Suppose we know that HamCycle is NP-complete. Show that it follows that HamPath is NP-complete by doing the appropriate polynomial time reduction. [1]

For both these decision problems, the input graph is undirected.

HamCycle = $\{< G > | G$ is a graph with edge set E and vertex set V, $|V| \geq 3$,

and G contains a Hamilton Cycle$\}$

HamPath= $\{< G > | G$ is a graph with edge set E and vertex set V, $|V| \geq 3$,

and G contains a Hamilton Path$\}$

ANSWER:

Proof. First we prove the following claim.

Claim: HamCycle $\leq_P$ HamPath.

Proof. Let HP be a decider for HamPath. Then we can build a HamCycle decider HC as follows:

HC = "On input $< G >$, where G is a graph:

1. For each edge (u, v) in G:
    - 1.1 add a new pendant vertex u' adjacent only to u, and add a new pendant vertex v' adjacent only to v, and call the resulting graph $G_{u,v}$
    - 1.2 run HP on in put $< G_{u,v}$.
        - if HP accepts, ACCEPT.
        - if HP rejects, pick another edge that has not been tested and go to 1.1.
2. If all edges have been tested and none has led to acceptance, REJECT."

[1] Stuck? For half marks, reduce any of the Hamiltonian decision problems we studied to any other, and make the appropriate "if-then" claim that the reduction demonstrates: "if ______ is NP-complete, then so is ______."

This is a *polynomial* reduction of HamCycle to HamPath because, if HP is polynomial, then the entirety of HC is also polynomial time: HP is called a polynomial number of times, and the remaining code outside of the calls to HP is also polynomial.

Why HC is a decider for HamCycle: A graph G has a hamiltonian cycle if and only if G has a hamilton path that starts and ends at two vertices, call them s and t, that are adjacent in G, which occurs if and only if there is a Hamilton path in $G + (s, s') + (t, t')$, where s' is a new vertex added that is only adjacent to s, and t' is a new vertex added that is only adjacent to t. □

Since HamCycle reduces in polynomial time to HamPath, and HamCycle is known to be NP-complete, then HamPath is also NP-complete.

□