Decidability, Recognizability, Undecidability.

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Prove true claims (decidable? undecidable? recognizable? unrecognizable?) about the following languages:

0. $H_{TM} = \{ \langle M, w \rangle \mid M$ is a TM that halts on string $w \}$
1. $A_{TM} = \{ \langle M, w \rangle \mid M$ is a TM that accepts string $w \}$
2. $NonE_{DFA} = \{ \langle D \rangle \mid D$ is a DFA and D accepts some string $\}$
3. $Acc\varepsilon_{PDA} = \{ \langle P \rangle \mid P$ is a PDA and P accepts $\varepsilon \}$
4. $RejectsSomeString_{TM} = \{ \langle M \rangle \mid M$ is a TM and M rejects some string $\}$.
5. $Loops_{TM} = \{ \langle M, w \rangle \mid M$ is a TM that loops on input $w \}$
6. $SixStates_{TM} = \{ \langle M \rangle \mid M$ is a TM and M has ≥ 6 states $\}$.

$NonE_{DFA} = \{ \langle D \rangle \mid D$ is a DFA and $L(D) \neq \emptyset \}$

Claim: $NonE_{DFA}$ is decidable.

Proof: Recall we showed that $E_{DFA} = \{ \langle D \rangle \mid D$ is a DFA, $L(D) = \emptyset \}$ is decidable; let X be a decider-TM for E_{DFA}.

We construct a TM Y that decides $NonE_{DFA}$ as follows:

Y clearly halts on all inputs because:

Y clearly accepts $\langle D \rangle$ iff iff $L(D) \neq \emptyset$ ▨

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts string } w \}$

Claim: A_{TM} is

Proof:

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts } w \}$.

Claim: A_{TM} is undecidable.

Proof: BWOC. $\exists$ A_{TM} is decided by some TM, call it W.

Then we can construct a TM V that decides SelfAcc, as follows:

V = "On input $\langle M \rangle$, where M is a TM:

1. Run W on $\langle M, \langle M \rangle \rangle$
2. If W accepts, ACCEPT.
   If W rejects, REJECT."

But Self Acc is undecidable!

$\therefore$ W does not exist, and A_{TM} is undecidable. ∎

A_{TM} is recognizable but not decidable.

6. SixStates_{TM}

$\text{RejectsSomeString}_{TM} = \{ \langle M \rangle \mid M$ is a TM that rejects some string $\}$

Claim: $\text{RejectsSomeString}_{TM}$ is undecidable.

Proof: BWOC. $\$ \ \exists$ a TM X that decides $\text{RejectsSomeString}_{TM}$.

Then we can construct a TM A that decides A_{TM}, as follows:

$A =$ " on input $\langle M, w \rangle$, where M is a TM, w is a string

1. Create a new TM M_w

   that a) erases its input

   b) runs M on w.

   c) '

⋮

leave the remainder as an exercise.

$E_{TM} = \{ \langle M \rangle \mid M \text{ accepts no strings} \}$.

Claim: E_{TM} is undecidable.

Proof: BWOC. Suppose $\exists$ a TM E that decides E_{TM}.

$A =$ " on input $\langle M, w \rangle$, where M is a TM, w a string.

1. Create a new TM M_w where

   $M_w =$ " 1. erase input
   2. run M on w. "

2. Run E on $\langle M_w \rangle$.
   - if E accepts, REJECT.
   - if E rejects, ACCEPT.

If A's input is $\langle M, w \rangle$ where M accepts w, then E rejects M_w, and A accepts $\langle M, w \rangle$.

If A's input is $\langle M, w \rangle$ where M either rejects or loops on input w, then M_w either rejects all inputs or loops on all inputs — but certainly accepts no inputs. Then E accepts $\langle M_w \rangle$ and so A rejects $\langle M, w \rangle$, as it should.

$\therefore$ A decides A_{TM}.

$\Rightarrow\Leftarrow$

$\therefore$ E_{TM} is not decidable.