Theorem 4.22

L is decidable $\Longleftrightarrow$ L is recognizable and $\bar{L}$ is recognizable

Proof: ($\Rightarrow$) Suppose L is decidable, and X is a decider for L.

Then X is also a recognizer for L.

Furthermore, we can construct a recognizer $\bar{X}$ for $\bar{L}$ as follows:

$\bar{X}$ = " on input $\langle w \rangle$,

1. Run X on w.
   - if X accepts, REJECT.
   - if X rejects, ACCEPT."

$\bar{X}$ always halts, since X is a decider.
$\bar{X}$ accepts iff X rejects, ie if $w \notin L$
i.e. when $w \in \bar{L}$. $\therefore$ $\bar{X}$ decides $\bar{L}$. ∎

($\Leftarrow$) Suppose L has a recognizer Y and $\bar{L}$ has a recognizer $\bar{Y}$.
Then we can construct a decider Z for L as follows:

Z = " on input $\langle w \rangle$

1. Run Y on $\langle w \rangle$ and run $\bar{Y}$ on $\langle w \rangle$, in parallel.
   - whichever one halts first, do the following as appropriate:
     - if Y accepts, ACCEPT.
     - if Y rejects, REJECT.
     - if $\bar{Y}$ accepts, REJECT.
     - if $\bar{Y}$ rejects, ACCEPT."

Note that Z will always halt, because one of Y or $\bar{Y}$ will have to halt and accept. Furthermore, we have programmed Z to accept iff $w \in L$, reject otherwise. ∎