

Note that we use “TM” to mean “Turing Machine”. The following languages are known to be **Undecidable**

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts string } w \}$.

$Halt_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that halts on string } w \}$.

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no strings} \}$.

$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2) \}$.

1. (10 marks) Show the the following language is **undecidable**:

$OneAccepts_{TM} = \{ \langle M_1, M_2, w \rangle \mid M_1 \text{ and } M_2 \text{ are TMs exactly one of them accepts } w \}$. For full marks, give a brief argument that shows your reduction works.

2. (10 marks) Show the the following language is **undecidable**:

$StateUse_{TM} = \{ \langle M, w, q \rangle \mid M \text{ is a TM, } \langle w \rangle \text{ is a string, and } q \text{ is a state in } M, \text{ and } M, \text{ when run on input } \langle w \rangle, \text{ visits state } q \text{ during its computation} \}$. For full marks, give a brief argument that shows your reduction works.

3. (a) (4 marks) Prove that the class of Turing-decidable languages is closed under intersection.
[Hint: “Let L_1 and L_2 be two Turing-decidable languages, decided by X_1 and X_2 respectively. Then we can construct a TM X that decides the language $L_1 \cap L_2$ as follows: ...”]

(b) (6 marks) Prove that the class of Turing-recognizable languages is closed under union.

4. Are the following languages recognizable? Prove your answer.

(a) (10 marks) $Reject_{any} = \{ \langle M \rangle \mid M \text{ is a TM, and there is a string } w \text{ that } M \text{ rejects} \}$.

(b) (10 marks) $OneAccepts = \{ \langle M_1, M_2, w \rangle \mid M_1 \text{ and } M_2 \text{ are TMs, and exactly one of } M_1 \text{ and } M_2 \text{ accepts } w. \}$.

5. (10 marks) Let L be a language that is enumerated by an Enumerator TM P_L .

(a) Is L recognizable? Prove your answer.

(b) Suppose there is also a transducer TM, call it T_L , that takes as input a string w , and outputs an integer r such that, if w is in L , then it must be printed by P_L within the first r strings printed. Is L decidable? Prove your answer.