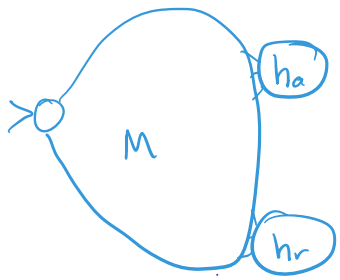
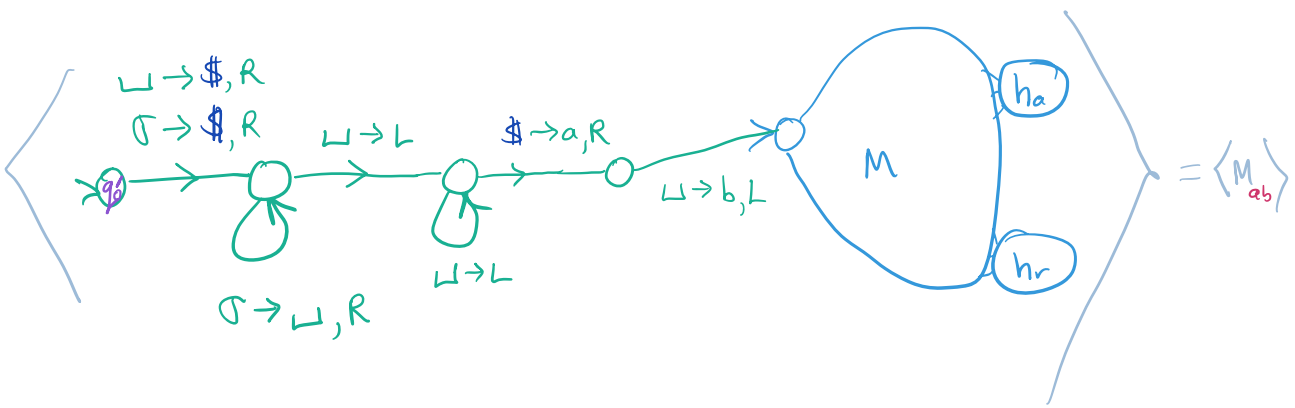
Can ∃ a TM that takes $\langle M, w \rangle$ and produces $\langle M_w \rangle$?

First, let's take an arbitrary M and a string w = ab and see what the result of that conversion would look like :

Given $\langle$ 

, ab $\rangle$ = $\langle M, ab \rangle$



Note that the above is the effect of the conversion for any M, and the specific string ab, but a "$\langle M, w \rangle$ to $\langle M_w \rangle$ converter TM" could obviously build the TM-encoding $\langle M_w \rangle$ for any M, w pair. It is also clear that such a conversion algorithm is

TM-computable — the converter-TM would add about $|w|+2$ states and the green-coloured transitions as above, and make the new q_0' be the new start state. \$ also needs to be added to the tape alphabet.

YES, $\exists$ a TM that takes $\langle M, w \rangle$ and produces $\langle M_w \rangle$.