E_{TM} is undecidable

$E_{TM} = \{ \langle M \rangle \mid M$ is a TM that accepts no strings $\}$

Claim: E_{TM} is undecidable.

Proof: BWOC. Suppose E_{TM} is decidable, and let X be a TM that decides it.

Then we can construct a TM Y that decides A_{TM} as follows:

Y = " on input $\langle M, w \rangle$, where M is a TM, and w a string

1. Construct a TM-description $\langle M_w \rangle$, where

   M_w = " on input $\langle s \rangle$, where s is a string

   1. Erase $\langle s \rangle$, and write $\langle w \rangle$ instead on input tape.
   2. Run M on the input." // i.e, on $\langle w \rangle$

2. Run X on input $\langle M_w \rangle$.
   - if X accepts, REJECT
   - if X rejects, ACCEPT. "

Note that all the instructions besides running X are do-able by a TM (and for notes on whether a TM can construct $\langle M_w \rangle$ if it is given $\langle M, w \rangle$, see another note in weekbyweek on the subject).

So, since X is a decider, so is .

What kind of TM is M_w?

If M accepts w then M_w accepts all strings.
ie $M_w \notin E_{TM}$, and Y accepts $\langle M, w \rangle$

If M does not accept w then M_w accepts No strings, .

ie $M_w \in E_{TM}$, and $\langle M, w\rangle$

So Y accepts $\langle M,w\rangle$ exactly if M accepts w, and
Y rejects $\langle M,w\rangle$ exactly if M does not accept w.

$\therefore$ Y decides A_{TM}.

$\Rightarrow\Leftarrow$ [A_{TM} is known to be undecidable].

$\therefore$ Y cannot exist, and E_{TM} is undecidable. $\blacksquare$