Assignment 4 CSCI 320 2025

1. [8 marks]

* marked

DiffOnString = { ⟨M_1, M_2, w⟩ | M_1 and M_2 are TMs, and exactly one of M_1, M_2 accepts w. }

Is DiffOnString decidable? Prove your answer.

Solution:

1 mark: Claim: DiffOnString is undecidable.

6 marks: Proof: BWOC. Suppose ∃ a TM X that decides DiffOnString. Then we can construct a TM Y that decides A_{TM} as follows:

Y = "On input ⟨M, w⟩, where M is a TM and w a string:

1. Construct a TM R that always rejects.
2. Run X on input ⟨M, R, w⟩.
   - if X accepts, ACCEPT.
   - if X rejects, REJECT."

1 mark: Y gives the correct answer, because X is a DiffOnString decider, so X only accepts ⟨M, R, w⟩ if M has a different acceptance-response to w than R ie if M accepts w.

; Y always halts, since all instructions are TM-doable except R, which always halts under the given assumption.

$\therefore$ Y decides A_{TM}

$\Rightarrow \Leftarrow$

Hence X cannot exist, and DiffOnString is undecidable. $\blacksquare$

2. [8 marks] $ALL_{TM} = \{\langle M\rangle \mid M$ is a TM that accepts all strings$\}$

Is ALL_{TM} decidable? Prove your answer

(1) Solution: Claim: ALL_{TM} is undecidable.

(6) Proof: [You can do a reduction from E_{TM} that "rewires" the input TM $\langle M\rangle$ by swapping h_a and h_r in M, but I will show another good option]

Reduction from A_{TM}. Let X be a TM that decides ALL_{TM}. Consider the following TM Y which, we will show, decides A_{TM}.

Y = "On input $\langle M, w\rangle$, where M is a TM, and w is a string

1. Construct a TM M_w, where M_w is:

   M_w = "on input $\langle x\rangle$, where x is a string:
   1. erase $\langle x\rangle$ from tape
   2. Write $\langle w\rangle$ on tape.
   3. Run M (i.e. on $\langle w\rangle$)"

2. Run X on input $\langle M_w\rangle$.
   - if X accepts, ACCEPT.
   - if X rejects, REJECT."

(1) Note that the TM M_w either

- accepts all strings (if M accepts w)
  ... in which case X accepts $\langle M_w\rangle$ and Y accepts $\langle M, w\rangle$
- or rejects all strings (if M does not accept w)
  ... in which case X rejects $\langle M_w\rangle$ and Y rejects $\langle M, w\rangle$

In each case, Y always halts and correctly decides if M accepts w.

$\therefore$ Y decides A_{TM}

$\Rightarrow\Leftarrow$

Hence X cannot exist, and DiffOnString is undecidable. $\blacksquare$

3. WriteSymb = { ⟨M, w, σ⟩ | M is a TM which, on input ⟨w⟩, at some point writes the symbol σ on the tape. }

[4 marks] a) Is WriteSymb recognizable? Prove your answer

[8 marks] b) Is WriteSymb decidable? Prove your answer.

Solution:

a) Claim: WriteSymb is recognizable.

Proof: WriteSymb is recognized by the following TM:

Z = "on input ⟨M, w, σ⟩:

1. Run M on input w.
   - if M ever writes σ onto the tape, ACCEPT."

b) Claim: WriteSymb is undecidable.

Proof: By reduction from A_{TM}. Let X be a TM that decides WriteSymb. Then we can construct a TM Y as follows:

Y = "on input ⟨M, w⟩, where M is a TM, w a string:

1. Introduce a new symbol, say #, to M's tape alphabet.
2. Alter M so that every transition to h_a writes # to the tape; call result M′
3. Run X on input ⟨M′, w, #⟩.
   - if X accepts, ACCEPT.
   - if X rejects, REJECT."

Note that M is not a transducer, so M′ has exactly same result on w as M. Note that M′ only writes # if M accepts w.

Also, Y accepts iff M' writes # iff M accepts w.

All instructions are TM-doable, so Y always halts.

$\therefore$ Y decides A_{TM}

$\Rightarrow\Leftarrow$.

$\therefore$ WriteSymb is undecidable. $\blacksquare$

* 4. BothAcceptSomeString $= \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs, and $\exists$ a string w that both M_1 and M_2 accept $\}$

a) [8 marks] Show that BothAcceptSomeString is recognizable.

b) [8 marks] Show that BothAcceptSomeString is not decidable.

c) [2 marks] Use a theorem from class to show that $\text{Intersection}_\emptyset$ is not recognizable:

$\text{Intersection}_\emptyset = \{\langle M_1, M_2\rangle \mid \nexists$ a string that both M_1 and M_2 accept $\}$

Solution:

a) (1 mark) Claim: BothAcceptSomeString is recognizable.

(7 marks) Proof: The following TM Z recognizes BothAcceptSomeString:

$Z =$ " on input $\langle M_1, M_2\rangle$ where M_1 and M_2 are TMs:

1. Let Σ_1 be M_1's alphabet, and Σ_2 be M_2's alphabet. Let $s_1, s_2, s_3, \ldots$ be the strings over the alphabet $\Sigma_1 \cap \Sigma_2$ in shortlex order.

2.1 Let $i = 1$.

2.2 Run M_1 and M_2 for i steps each on each string $s_1, s_2, \ldots s_i$.

If both accept some string, ACCEPT.

otherwise, $i = i+1$, and go to 2.2."

If $\exists$ a w that both M_1 and M_2 accept, Z will eventually find it and accept, after a finite number of steps. If $\nexists$ such a w, Z will run forever (ie loop).
$\therefore$ Z is a recognizer for BothAcceptSomeString. $\boxed{}$

b) (1 mark) Claim: BothAcceptSomeString is undecidable.

(7 marks) Proof: By reduction from A_{TM}. Let X be a decider for BothAcceptSomeString. Then we can construct a TM Y that decides A_{TM} as follows:

Y = "on input $\langle M, w \rangle$, where M is a TM, w a string
1. Construct a TM A_w that accepts only the string w and rejects everything else.
2. Run X on input $\langle M, A_w \rangle$.
   - if X accepts, ACCEPT.
   - if X rejects, REJECT."

Since A_w only accepts w, M and A_w are only accepted by X if M accepts w.
Hence Y accepts iff M accepts w, and rejects therwise.
$\therefore$ decides A_{TM}.

$\Rightarrow\Leftarrow$

$\therefore$ BothAcceptSomeString is undecidable $\boxed{}$

c) (2 marks)

"BASS"

BothAcceptSomeString is undecidable.

$\therefore$ by Theorem 4.22, at least one of BASS, $\overline{\text{BASS}}$ is unrecognizable.

BASS is recognizable.

$\therefore$ $\overline{\text{BASS}}$ is not recognizable.

Note that $\overline{\text{BASS}}$ = $\text{Intersection}_{\emptyset}$.

$\therefore$ $\text{Intersection}_{\emptyset}$ is not recognizable. $\blacksquare$