Thm 4.11 A_{TM} is undecidable.

Proof was done in tutorial.

We reduced Self-Acc to A_{TM}. ← trying to prove is hard.

language language

Known hard

Black Box A_{TM}-decider

Allows us to construct a Self-Acc decider that calls the Black-Box A_{TM} decider as a subroutine

Algorithm reductions are common.

Eg. To tell if a graph is connected

- call $\boxed{\text{Shortest Paths}}$
- check that every node has non-∞ path length to every other node.

reduces connectivity to shortest paths.

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM that accepts no strings} \}$.

Theorem: E_{TM} is undecidable.

Proof: We reduce A_{TM} to E_{TM}.

I.e. we assume (BWOC) that E_{TM} is decided by a TM, call it X (decides E_{TM}).

We construct a TM Y as follows: [idea: Y decides A_{TM}]

Y = "on input $\langle M, w \rangle$ where M is a TM, w a string:

A_{TM} decider

1. Construct a new TM, M_w, that works as follows:

   M_w 1. Erase the input, and write w on the tape.

   M_w 2. Run M on the input tape contents, w.

2. Run X on $\langle M_w \rangle$
3. If X accepts, REJECT.
   If X rejects, ACCEPT"

What does M_w do on all inputs if M accepts w?

- M_w accepts all inputs hence $L(M_w)$ is not empty and X rejects. (+ vice versa).

$\therefore$ Y is a decider for A_{TM}. $\Rightarrow\Leftarrow$.

$\therefore$ E_{TM} is undecidable. $\blacksquare$

"M"

square(x)
  return (x * x);

"M_w" $(w = 10)$.

square10 (x)
  return square (10);

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \}.$$

Theorem: EQ_{TM} is undecidable

Proof: We reduce the decidability of EQ_{TM} to the decidability of E_{TM}

Suppose $\exists$ a decider W for

We construct a decider T for as follows:

T = "on input $\langle M \rangle$, where M is a TM:

1. Construct
2.
3.

Since W is a decider for EQ_{TM}, then clearly T
- always halts
- accepts $\langle M \rangle$ iff M is equivalent to $M_\emptyset$ ie $L(M) = \emptyset$.

$\therefore$ T decides E_{TM}

$\Rightarrow\Leftarrow$

$\therefore$ W cannot exist, and EQ_{TM} is undecidable. $\blacksquare$

Let $NotA_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that does not accept } w\}$

Note: $NotA_{TM} = \overline{A_{TM}} \cap \{\langle M, w\rangle \mid M \text{ is a TM, } w \text{ is a string}\}$.

It is common to just refer to $NotA_{TM}$ as $\overline{A_{TM}}$, but is technically not correct.

Theorem 4.23 $NotA_{TM}$ is not recognizable

Proof: BWOC. Suppose $NotA_{TM}$ is recognized by some TM $\bar{A}$.

This tells us what the hard part of A_{TM} is ...

(though we already knew)

A_{TM} – YES! (ACCEPT) answers – we get those

$\overline{A_{TM}}$ – NO! (Reject or Loop) answers – we don't get them all.