Aside: $E_{DFA} = \{ (\{q_0\}, \{a,b\}, \{\}, q_0, \emptyset), (\{q_0\}, \{a,b\}, \ldots) \ldots$

Decidability (cont'd)

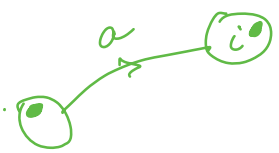
$E_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA, } L(A) = \emptyset \}$

Thm 4.4 E_{DFA} is decidable.

Proof: E_{DFA} is decided by TM T where:

$T =$ "on input $\langle A \rangle$, where A is a DFA:

1. Mark the start state of A.
2. Until no new states get marked, (ie in this iteration.) do:
   - 2.1 Mark any unmarked state that has a transition into it from a marked state.
3. If no accept state is marked, ACCEPT. Otherwise, REJECT."



The loop in step 2 will terminate
- # iterations cannot be greater than number of states in A.

terminates

Correctness:

$\exists$ a string accepted by A $\iff$ $\exists$ a directed path from start state to an accept state in A.
$\iff$ an accept state gets marked by T. ▨

Correctness

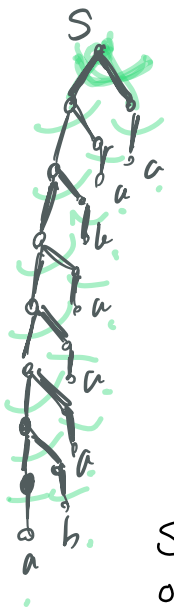
$EQ_{DFA} = \{ \langle A, B \rangle \mid A \text{ and } B \text{ are DFAs, and } L(A) = L(B) \}$

- tutorial.

$A_{CFG} = \{ \langle G, w \rangle \mid G \text{ is a CFG, that generates string } w \}$

Theorem: A_{CFG} is decidable.

Proof: The TM S decides A_{CFG}:



S = "on input $\langle G, w \rangle$ where G is a CFG and w a string:

1. Convert G to CNF.
2. List all derivations of length $2n-1$ where $|w| = n$ (or if $w = \varepsilon$, where $n = 1$).
3. If any of the derivations derive w, ACCEPT. otherwise, REJECT."

S always halts because $\exists$ a finite # of derivations of length $2n-1$. It is correct, because it only accepts if it finds a derivation of w. ▨

$E_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset \}$.

$\emptyset \neq \{\varepsilon\}$

eg
$D \to ABC$
$A \to Aa \mid BB$
$B \to bB \mid b$
$C \to Dc$

Theorem 4.8 E_{CFG} is decidable.

Proof: E_{CFG} is decided by Tm R:

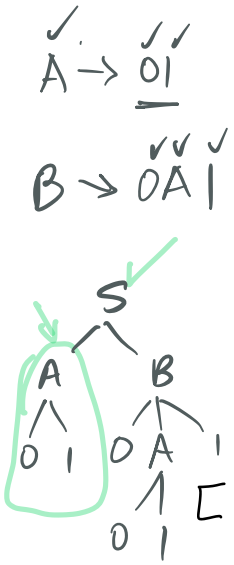
R = " on input $\langle G \rangle$, where G is a CFG:

1. mark all terminals in G (on RHS of rules)

2. Repeat until no new variables are marked:

   2.1 Mark any variable A where A has a rule $A \to \sigma_1 \sigma_2 \cdots \sigma_n$ (all marked).

3. If start symbol is not marked, ACCEPT. Otherwise REJECT. "



[we argued as to why this is correct; it terminates because at least one variable is marked at each iteration – we run out of variables in finite # of steps]

Eg R accepts $(\{S\}, \{a,b\}, \{(S \to SS)\}, S)$

R rejects $(\{S\}, \{a\}, \{(S \to a)\}, S)$

$S \to AB$

$A \to 0A1 \mid 0A$

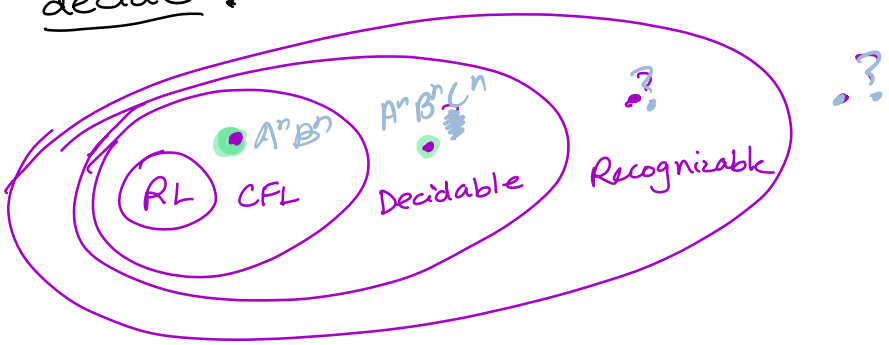
$B \to 0A1 \mid 1A0$

REJECT.

Chapter 4.2 Undecidability

∃? a problem / language that a computer (TM) cannot recognize?

∃? a problem / language that a computer (TM) cannot decide?



Claim: The language TMencodings $= \{\langle M \rangle \mid M \text{ is a TM over } \Sigma\}$ is countably infinite.

Proof: Sketch: 1. Each TM has an encoding over the alphabet $\Gamma = \{(,), ,, 0, 1, \{, \}\} \cup \Sigma$.

2. The strings over Γ are enumerable (eg in shortlex order)
3. Remove the strings that are NOT a TM encoding, and you are left with a loose enumeration of TM encodings.

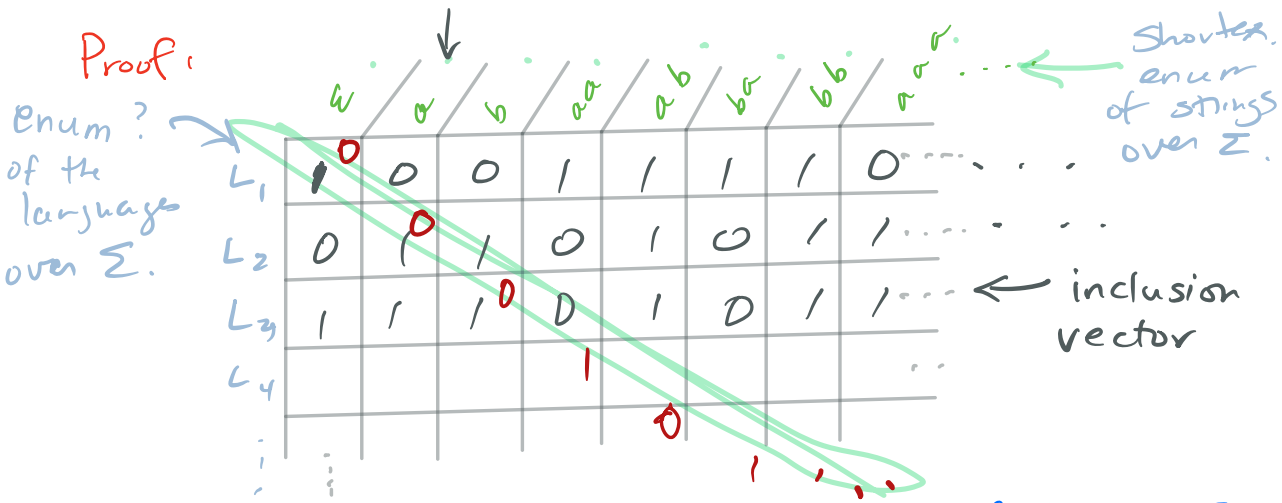
Theoretically, we could enumerate all the TM encodings in shortlex order. $\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle, \ldots$

correspondingly we have a list

$M_1, M_2, M_3, \ldots$ ▨

Claim: The set of languages over Σ is uncountable.

Proof:

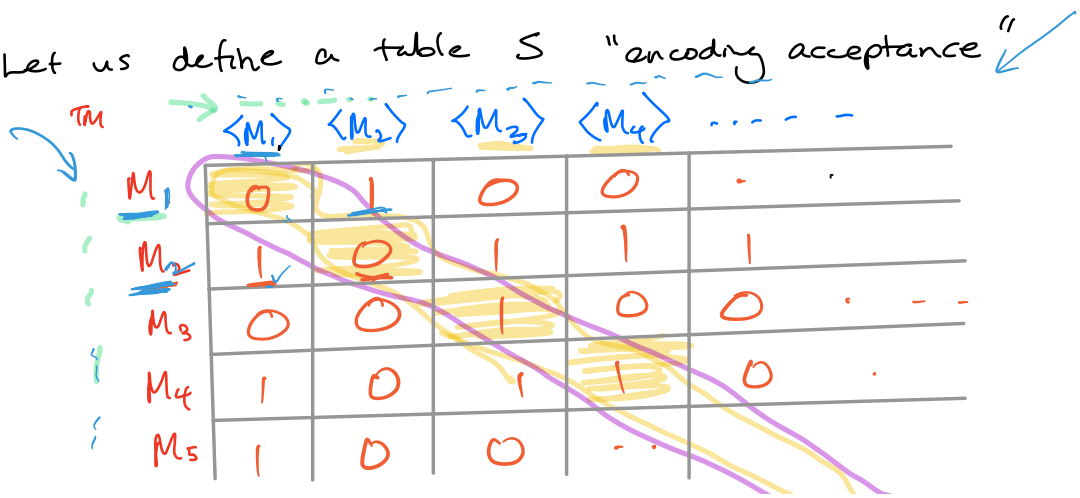


Diagonalization shows that the number of Languages over Σ is uncountable! ▨

Corollary: ∃ languages that are not Turing recognizable.

Do ∃ languages that are recognizable but not decidable?

Let us define a table S "encoding acceptance"



Let $D[i] = S[i][i]$

if $D[i] = \begin{cases} 1 & \text{then } M_i \text{ accepts } \langle M_i \rangle \\ 0 & \text{then } M_i \text{ does not accept } \langle M_i \rangle \end{cases}$

$SelfAcc = \{ \langle M \rangle \mid M \text{ is a TM that accepts } \langle M \rangle \}$

SelfAcc is the language of TMencodings that are encodings of TMs that would halt-accept when run on their own encoding as input.

$SelfNotAcc = \{ \langle M \rangle \mid M \text{ is a TM that does not accept } \langle M \rangle \}$

Note that $SelfNotAcc = \overline{SelfAcc} \cap TM\text{encodings}$

We have seen that TMencodings is decidable.

The following theorem is useful:

Theorem: The class of decidable languages is closed under complement, $\cap$, $\cup$.

Proof: $\cup$, $\cap$ left as an exercise.

Complement: let L be any decidable language, and let M be a TM that decides L. Then we can construct a new TM $\bar{M}$ that decides $\bar{L}$ as follows:

$\bar{M}$ = " on input $\langle w \rangle$

1. Run M on $\langle w \rangle$;
   if M accepts, REJECT.
   if M rejects, ACCEPT."

1 Always terminates.

2. Correct.

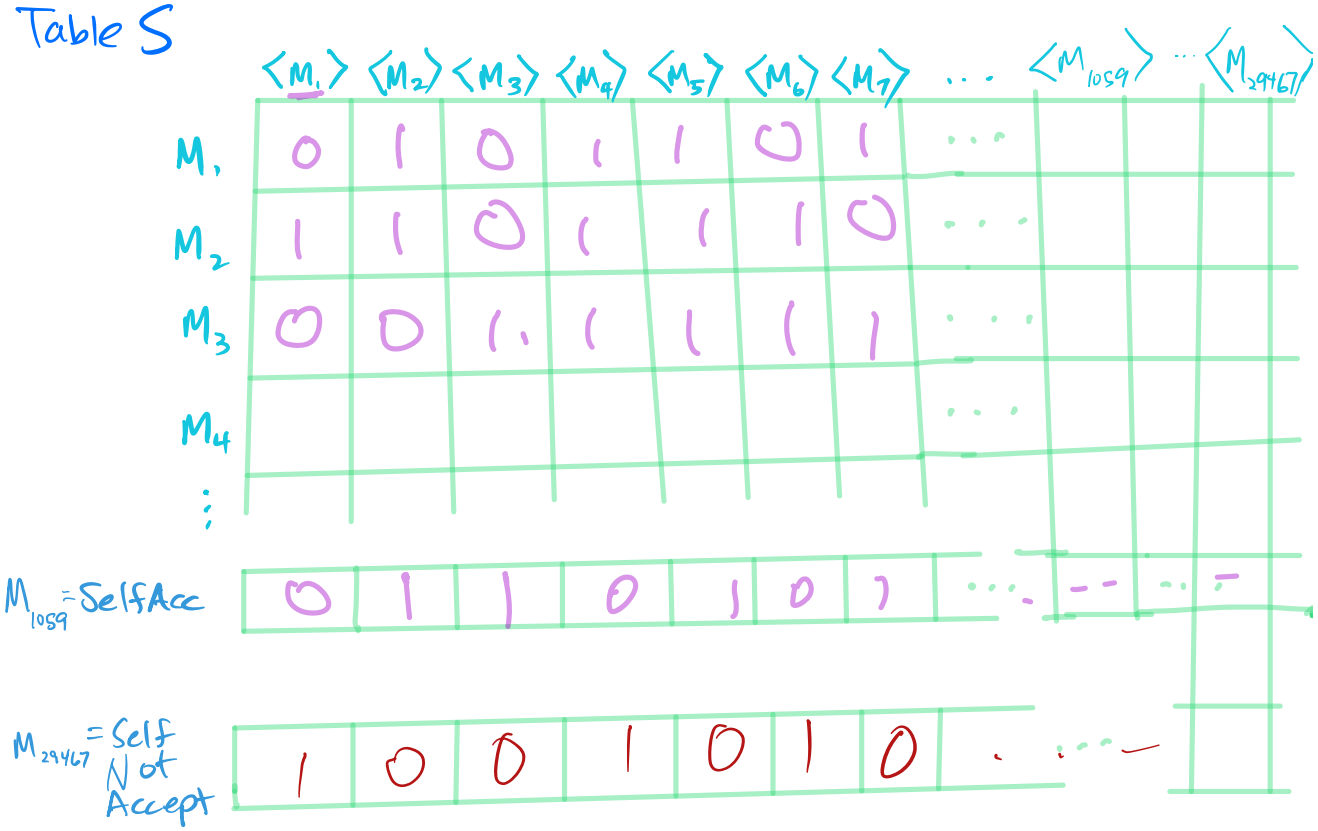
Mar 21

Theorem: SelfNotAccept is not decidable

Proof: We have seen that the TM encodings are enumerable, for example in shortlex order.

Then we can construct the following table:

Table S



Each TM M_i can be fed input $\langle M_j \rangle$, and M_i either accepts or does not.

$SelfAcc = \{ \langle M_i \rangle \mid S[i][i] = 1 \}$.

and the inclusion vector of SelfAcc is the diagonal of Table S.

The inclusion vector of SelfNotAcc is the "bit flip" of the diagonal of Table S.

↑ 0→1
1→0

But then where is TM SelfNotAccept in our enumeration?

It's inclusion vector is the flip of the diagonal, so this inclusion vector is NOT in the table itself — it is not a TM that can exist!

Another way of putting it is the following:

Theorem 4: Self Not Accept is not decidable.

Proof: BWOC. $\nexists$ SelfNotAccept is decided by some TM X.

If X accepts $\langle X \rangle$, then

$\Rightarrow$ X is self-accepting

$\Rightarrow \langle X \rangle \notin$ SelfNotAccept

$\Rightarrow$ X does not accept $\langle X \rangle$

$\Rightarrow\Leftarrow$

If X does not accept $\langle X \rangle$

$\Rightarrow$ X is not self-accepting

$\Rightarrow \langle X \rangle \in$ SelfNotAccept

$\Rightarrow$ X accepts $\langle X \rangle$

$\Rightarrow\Leftarrow$

Either way, there is a contradiction.

$\therefore$ X does not exist, and SelfNotAccept is undecidable. ▨

Theorem: SelfAccept is not decidable.

Proof: BWOC. $\nexists$ SelfAccept is decided by some TM Y.

Then we can construct a TM X that decides SelfNotAccept as follows:

X = "on input <M> where M is a TM:

1. Run Y on <M>
   - if Y accepts, REJECT.
   - if Y rejects, ACCEPT."

X clearly returns the opposite answer to Y, accepting TM-encodings that reject themselves, and rejecting those that accept their own encodings — i.e., X decides SelfNotAcceptance, which contradicts Thm 4, above.

Note: we skipped the part where we reject the input if it is not a proper TM encoding. Perhaps it should read...

X = "on input ⟨w⟩:

0. Run T on ⟨w⟩, where T is a TM that decides TM encodings.
   If T rejects, REJECT.
   otherwise, go to step 1.

1. run Y on ⟨w⟩, (where we now know ⟨w⟩ = ⟨M⟩ for some TM M).

   ⋮

   etc. Just like the proof we gave above. "

It is our practice to just represent the above step 0 as "on input ⟨M⟩, where M is a TM"