Chapter 2.3 Non-Context-Free Languages.

Not all languages can be recognized by a PDA.

Consider $\{a^n b^n c^n \mid n \geq 0\}$ = "$A^n B^n C^n$"

Imagine a PDA that recognizes $A^n B^n C^n$.

It might push every a onto the stack,

then pop & match the a's with the b's.

But then what?

That's not a proof that no such PDA can do the job,

but attempting to construct such a PDA will lead to

recognizing some limitations of PDAs.

To prove there is no such PDA, we need...

Theorem 2.34 Pumping Lemma for CFLs.

$\forall$ CFL A, $\exists$ integer p such that,

$\forall s \in A$, $|s| \geq p$, $\exists\, u, v, x, y, z \in \Sigma^*$ where

$$s = uvxyz$$

where: 1. $uv^i x y^i z \in A$

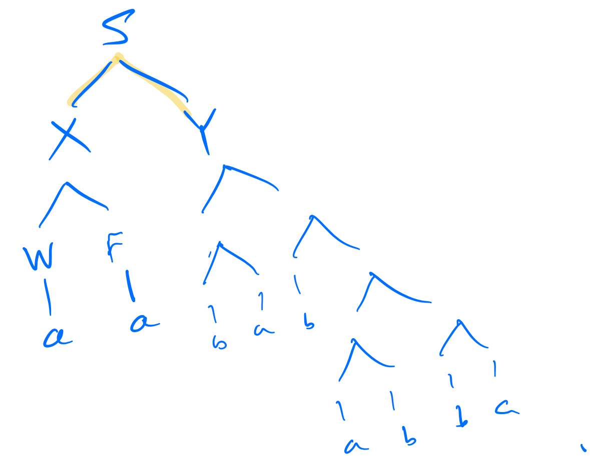
2. $|vy| > 0$

3. $|vxy| \leq p$

Proof idea: Let A be a CFL,

Let G be a CFG in CNF. for A

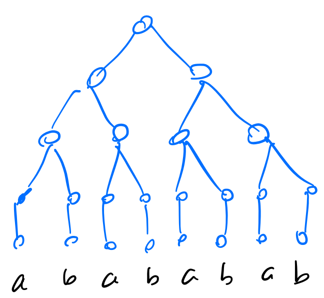
Then all derivation trees in G are binary trees (except at leaf).

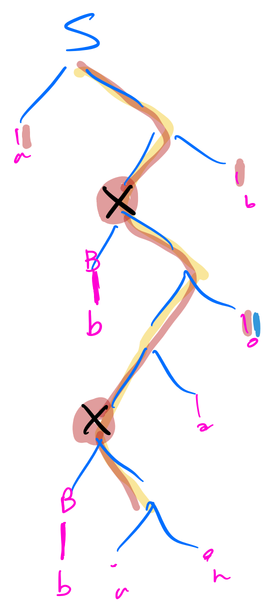


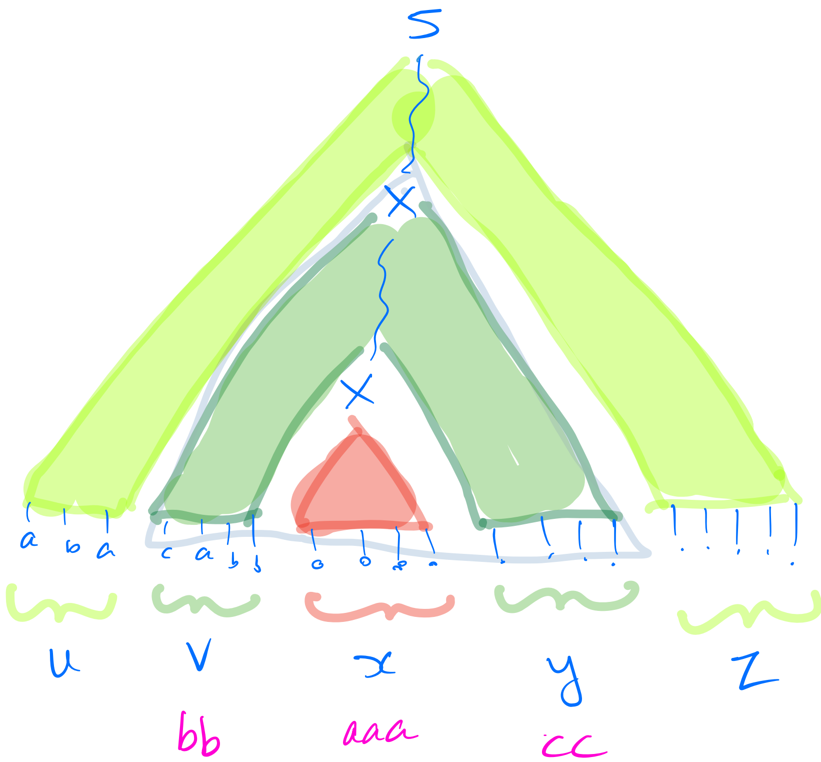
Then every string of length $\geq p$ must have height $\geq$ $\log_2 p + 1$.

If $\log_2 p + 1 >$ number of variables in G.

then the longest path in the derivation must reuse variables.



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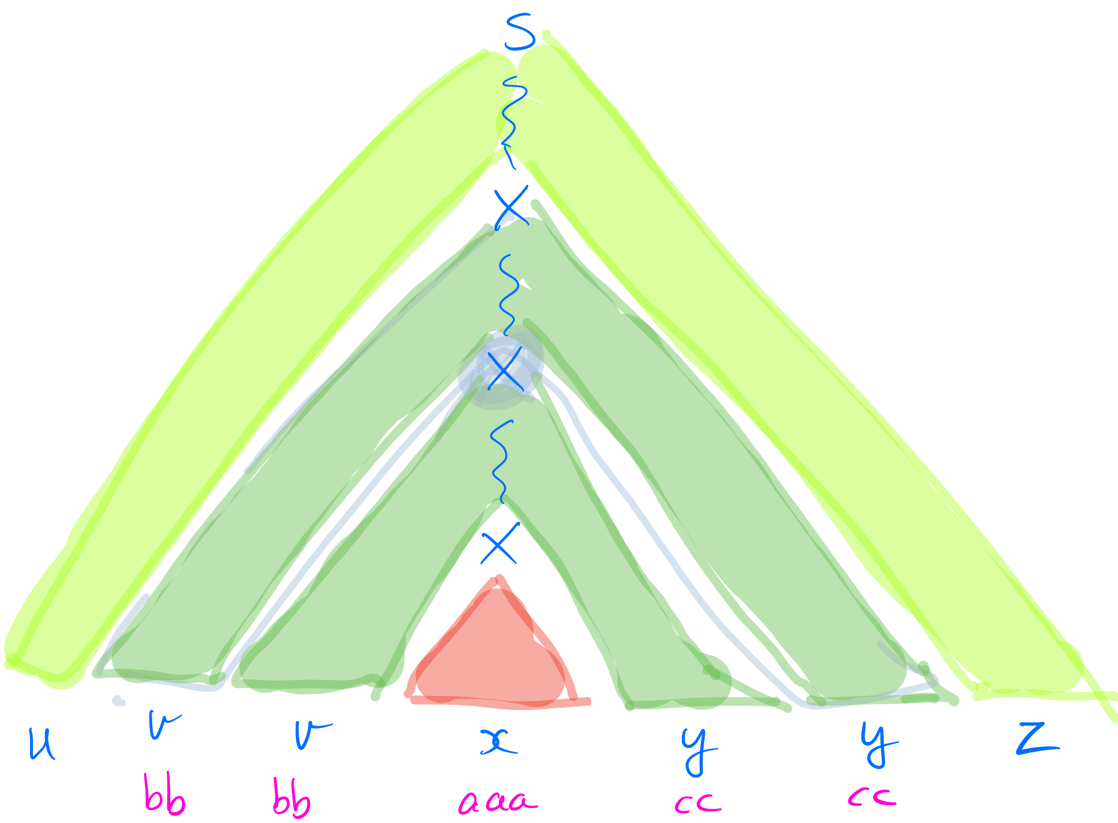
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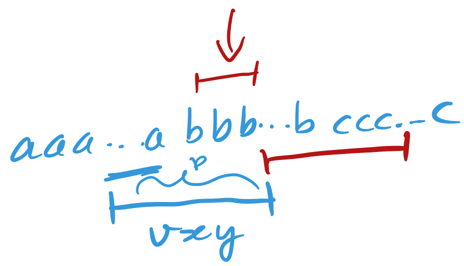
$\exists$ a derivation for uv^ixy^iz $\forall i = 0, 1, 2, \ldots$

$X \to \underbrace{aXYbaZ}_{b}$ b-ary.

$\boxed{/}$

Theorem: $A^nB^nC^n$ is not CF. $\{a^nb^nc^n \mid n \geq 0\}$

Proof: Suppose it were, and let p be its pumping constant.



Let $w = a^p b^p c^p$.

$w \in A^nB^nC^n$ and $|w| \geq p$, so

$w = uvxyz$ where $|vy| > 0$, $|vxy| \leq p$ ① ②

and $uv^ixy^iz \in A^nB^nC^n$ $\forall i \geq 0$, by P.L.

But ① and ② imply that v, x, y are such that:

Case 1 – if v contains a's then vxy does not contain c's (because a's are distance p from c's)

$\therefore$ pumping up yields more a's then c's.

Case 2 – if v does not contain a's, then pumping up yields more b's or more c's than a's. $\boxed{/}$

Other languages that are not Context-Free:

$$\{ww \mid w \in \{a,b\}^*\}$$

$$\{w \in \{a,b,c\}^* \mid \#_a(w) = \#_b(w) = \#_c(w)\}$$

$w = aabbaabbabab.$

p

$aabbbaabbbbcaabc.$

$v \quad y$

$p.$

Claim: $L = \{w\#w \mid w \in \{a,b\}^*\}$ is not CF.

Proof: BWOC. S L is CF and let p be its pumping constant.

Consider the string $a^p b^p \# a^p b^p$ which has length at least p so, by P.L.,

$a^p b^p \# a^p b^p = uvxyz$ where

$|vxy| \le p$ and $uv^i x y^i z \in L \ \forall i$

case 1: $\#$ is in v or y.

$\Rightarrow$ pumping down leads to a string with no $\#$.

$\Rightarrow\Leftarrow$ (not in L).

case 2: $\#$ is in x

$\Rightarrow$ v is b^t for some t

and $y = a^r$ for some r.

$r+t > 0$.

$\Rightarrow$ pumping up: more b's before $\#$ than after,

or: more a's after $\#$ than before.

$\Rightarrow\Leftarrow$

case 3: # is not in vxy.

Pump up once.

$\Rightarrow$ length of string before # is different from the length of string after #.

$\Rightarrow\!\Leftarrow$

$\therefore$ L is not CF. $\blacksquare$