Chapter 4 Turing Machines

Mar 3, 2026

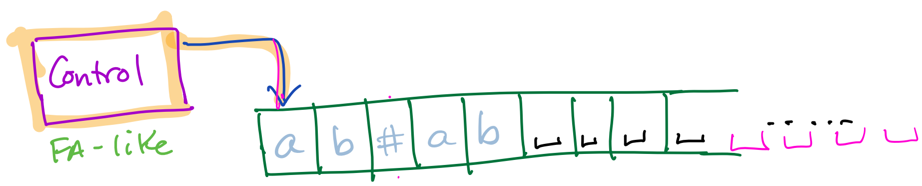
FA – good for small amount of memory

PDA – good for tasks use LIFO memory.

But we want to explore general computability, and neither FA nor PDA can even recognize $\{a^n b^n c^n \mid n \geq 0\}$ or $\{w\#w \mid w \in \{a,b\}^*\}$.

Turing Machines (TMs)

- Alan Turing in 1936
- FA with a one-way-infinite tape, R/W
  - tape head can move L and R.
  - ∃ an accept state and a reject state – halt



Formal Def$^{\underline{n}}$ of a TM.

Def^n 3.3 A Turing Machine (TM) is a 7-tuple.

$$(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$$

Where: 1. Q is a finite set of states.

2. Σ is input alphabet, $\sqcup \notin \Sigma$

3. Γ is tape alphabet, $\sqcup \in \Gamma$, and $\Sigma \subseteq \Gamma$

4. $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$ is the transition function.

5. q_0 is start state

6. q_a is ACCEPT state

7. q_r is REJECT state.

A TM computes as follows:

it starts with: – input string occupying all leftmost cells of tape, up to leftmost blank.

– tape is all "$\sqcup$" after the input string.

– tape head is on leftmost cell.

Start up M ...

at each step M: – reads cell under current head pos.

– is in a given state

– if state is q_a – Halt & ACCEPT

– if state is q_r – Halt & REJECT

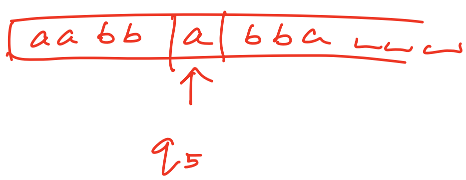
otherwise, based on state, tape cell contents:

– write a symbol to current cell

– Move L, R, S.

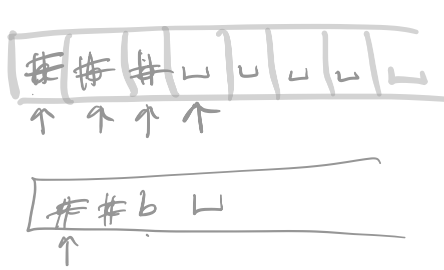
– change state.

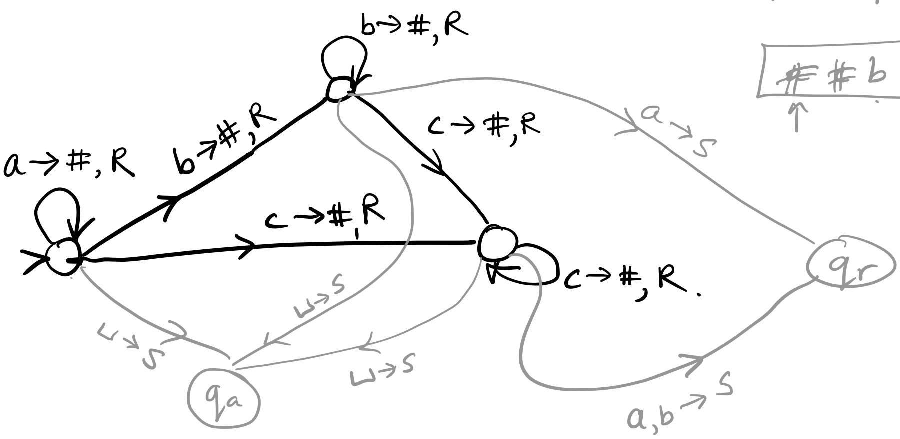
Note: if M is on leftmost cell and is supposed to move L, it just stays.



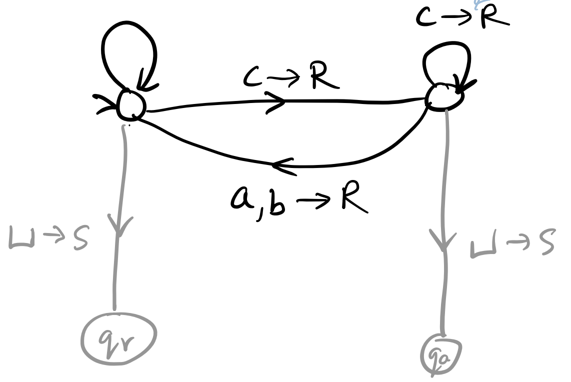
A configuration of a TM is all info necessary to say how rest of comput'n will proceed.

TM for $L(a^* b^* c^*)$:



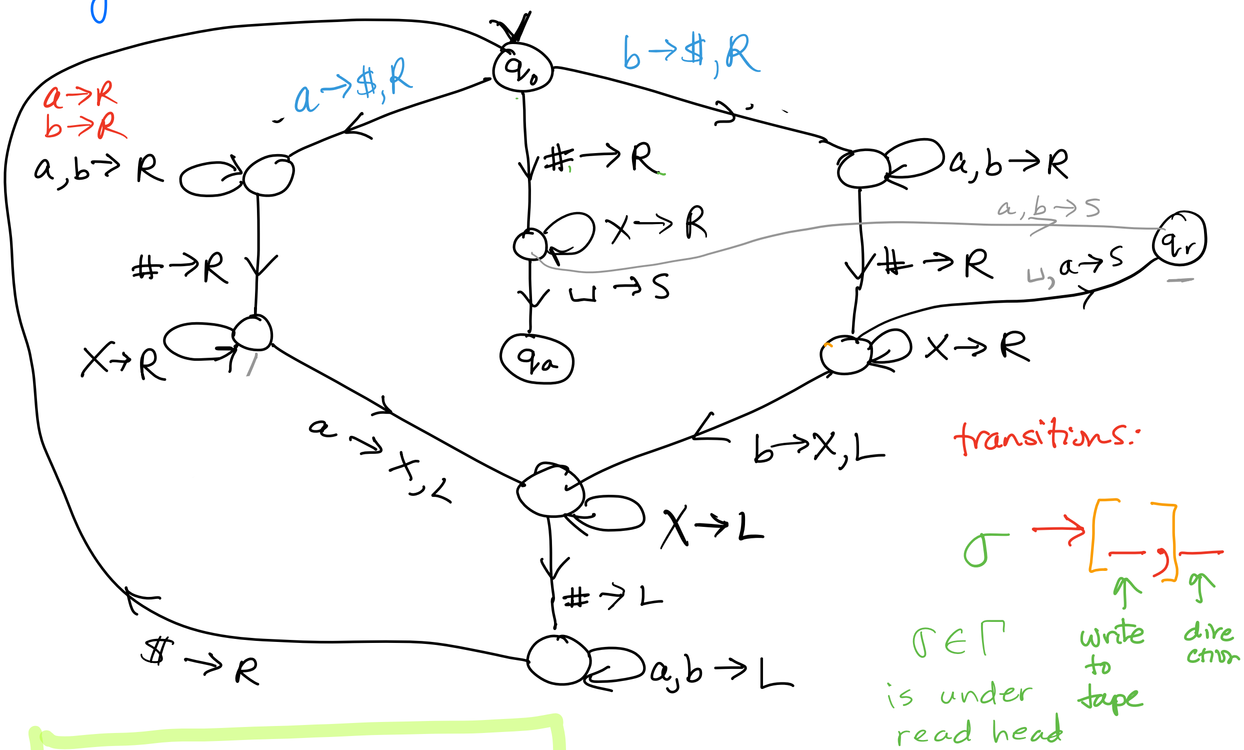


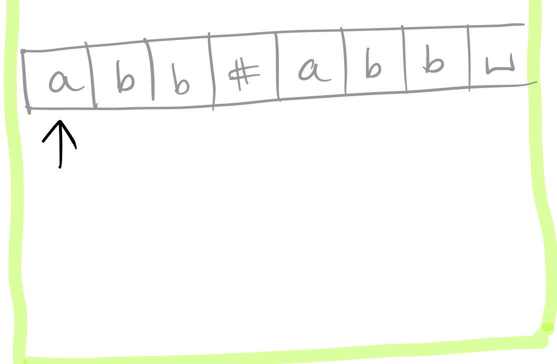
TM for "ends in c", $\Sigma = \{a, b, c\}$

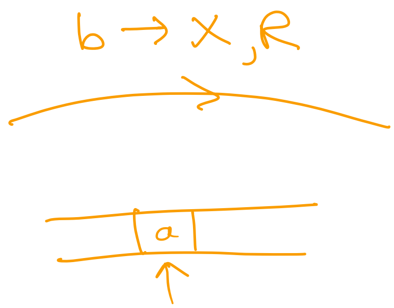


Note: c → R is same as
c → c, R

Eg TM that recognizes $\{w\#w \mid w \in \{a,b\}^*\}$



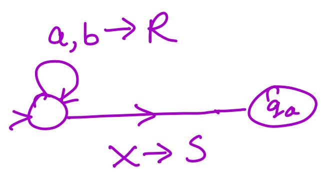




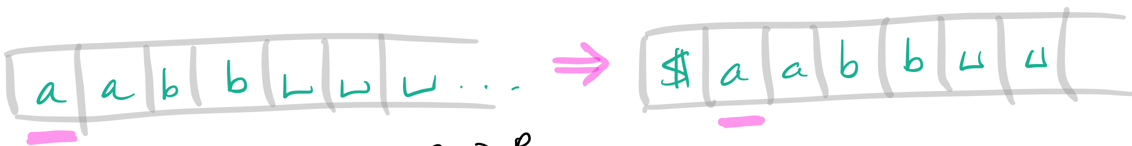
Every TM can be shown as a diagram in this way.

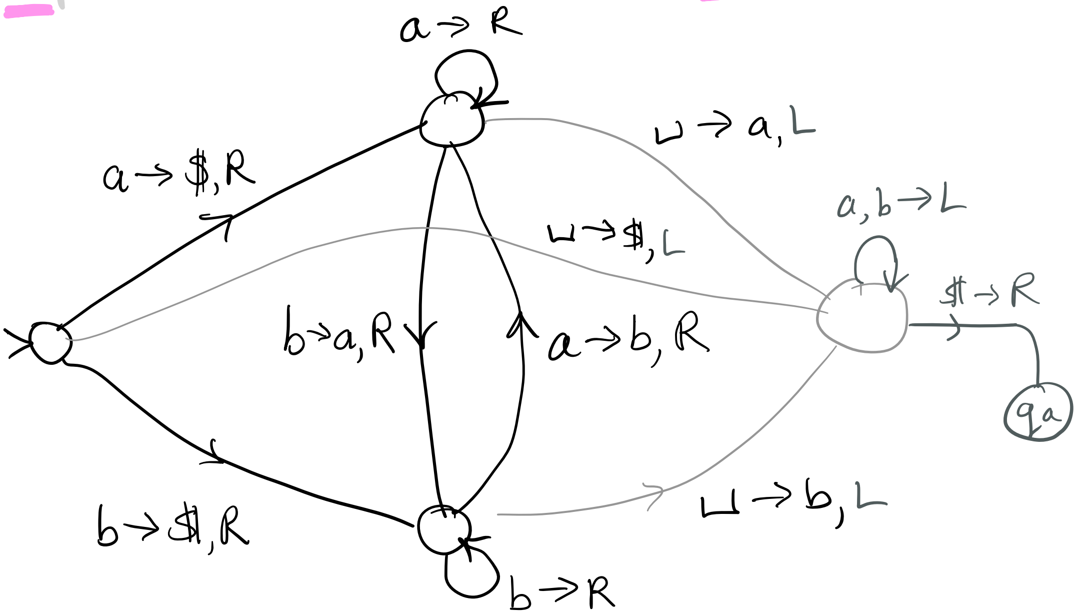
That said, they quickly become large and ungainly if they are doing anything complex. So we shift to a higher level description of a TM. — but we could draw the diagram if we wanted to.

eg. high-level – "scan R to first X and stay" for some $X \in \Gamma$



eg high-level description: "Shift tape contents R one cell, and insert \$ at leftmost cell" $\Sigma = \{a,b\}$





$M_{\$\,\text{ShiftR}}$ $\Sigma = \{a,b\}$

Such a TM is useful as a "function call" by a bigger TM. It transforms the tape contents — in essence, Mar 5

A transducer TM computes a function

$f: \Sigma^* \to \Gamma^*$

input = original tape contents

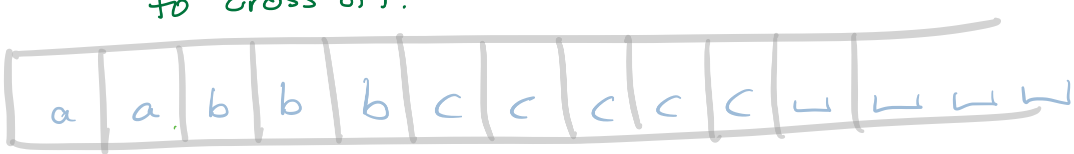
output = tape content after executing the transducer TM.

Eg of a high-level description of a TM.

$C = \{ a^i b^j c^k \mid i \times j = k \quad i, j, k \geq 1 \}$

M = " On input string w:

0. Shift-R the input string, leaving \$ on leftmost cell.
1. Scan R to determine if it is a member of $a^+ b^+ c^+$ — REJECT if not of this form.
2. Return head to leftmost cell.
3. Cross off an a and scan right to first b.
   Shuttle between b's and c's,
   B-ing out a b and X-ing out a c each time.
   If c's end early — REJECT.
4. [No more b's]:
   Restore B's as b's and go to 3, if ∃ an a to cross off.

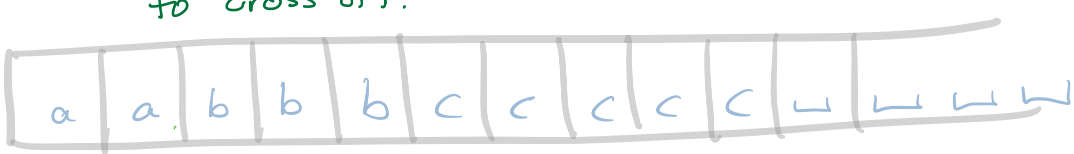


Eg of a high-level description of a TM.

$C = \{ a^i b^j c^k \mid i \times j = k \quad i, j, k \geq 1 \}$

M = " On input string w:

1. Scan R to determine if it is a member of $a^+ b^+ c^+$ — REJECT if not of this form.
2. Return head to leftmost cell.
3. Cross off an a and scan right to first b.
   Shuttle between b's and c's, X-ing out a b and a c each time.
   - if b's end early — REJECT
   - if c's end early — REJECT.
4. Restore X'd-out b's and go to 3, if ∃ an a to cross off.

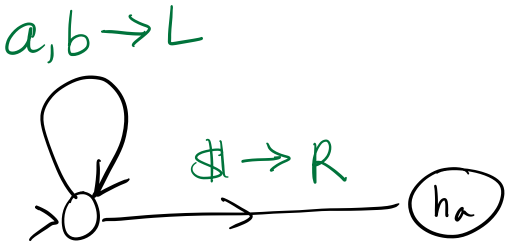


- If all a's have been crossed off,
  Check that all c's have been X-ed off.
  If yes - ACCEPT
  no - REJECT."

How do we return to leftmost cell?

We use $M_{\$ \text{shift}}$ to insert \$ onto leftmost cell (assuming \$ is not used elsewhere in the TM - ie introduce a symbol used only for this special purpose).

Then any time the high-level description says "move to leftmost cell", we do this:



first \$ encountered

"$L_{\$} R$" = "Move L to \$, then R once"

Next time: How does a TM "restore the <u>X-ed</u> out b's".

accepts if first letter is a , rejects o.w.

$a^* b^*$

$a^n b^n$

A missing transition means "hang" – another way of rejecting.

To "X-out b's" that may need to be restored:

Eg: A transducer TM that computes f where

$$f(w) = \begin{cases} \$w & \text{if } \#_a(w) \leq \#_b(w) \\ \$ & \text{otherwise} \end{cases}$$

R

Idea:

a	b	b	b	a	b	⊔	⊔	⊔	...		

1. Insert \$ to mark leftmost cell, Shift-R the cell contents, using $M_{\$ShiftR}$:

2. Scan L to \$.

3. Scan R to "a" or "␣"
- if "a", write "A",
- if "␣", go to 9.

4. Scan L to $
5. Scan R to "b", and write "B"
if no "b", go to 11
6. Scan L to $
7. Go to 3.
8. Scan L to $

Restore 9. Scan R to ␣, and, going L,
for each "A", write "a"
for each "B", write "b"
- until reach "$"; halt.

Erase 11. Scan R to ␣; move L.
12. Scan L, overwriting each "A", "B" "a", "b" with "␣"
When reach "$", stay and halt.