Using The Pumping Lemma for CFLs

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Claim: $L = \{ w\#w \mid w \in \{a,b\}^* \}$ is not CF.

Proof: BWOC. $\exists$ L is CF. Then it has a pumping constant — call it p.

Consider the string $w = a^p b \# a^p b$

Note that $|w| \geq p$ and $w \in L$, so by P.L. for CFLs,

$w = uvxyz$ where $u, v, x, y, z \in \{a, b, \#\}^*$, and

① $|vxy| \leq p$

② $|vy| > 0$

③ $uv^ixy^iz \in L \;\forall\, i \geq 0$

Case 1. vy contains $\#$

$\Rightarrow$ pumping down makes $\#$ disappear, so the resulting string cannot be $\in L$.

$\Rightarrow\Leftarrow$

Case 2. v is before $\#$, y is after $\#$, both non-empty

$\Rightarrow$ v is among first block of b's

and y is not in 2nd block of b's, by ②.

$\Rightarrow$ pumping up once increases b's before $\#$

while not increasing them after $\#$

- resulting string is not in L

$\Rightarrow\Leftarrow$

Case 3. v and y are on same side of $\#$, or one is empty.

$\Rightarrow$ pumping up yields a string that has more letters on one side of $\#$ than the other side.

$\Rightarrow$ resulting string is not $\in L$.

$\Rightarrow\Leftarrow$.

All cases lead to contradiction. $\therefore$ L is not CF. ▨

Theorem: The language-class CFL is closed under

$\cup$ $\quad\cdot\quad$ $*$ $\quad$ Reverse $\quad$ Letter-Substitution

Proof: By construction. Exercise for the student.

Hint: use CFG's. If helpful, you can assume the grammar is in CNF.

Theorem: The language-class CFL is not closed under

$\cap$ $\quad$ Complement $\quad$ Difference

Proof: We will show that CFL is not closed under $\cap$.

$\cap$ The following two languages are CF:

$$\{a^n b^n c^m \mid n, m \geq 0\} \qquad \{a^j b^i c^i \mid i, j \geq 0\}$$

However, their intersection is

$$\{a^n b^n c^n \mid n \geq 0\} = A^n B^n C^n,$$

which we have already shown to be NOT Context-Free. ∎ ($\cap$

Complement: If the class CFL were closed under complement (and we know it is closed under union), then it would also be closed under $\cap$, by De Morgan's:

$$L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$$

∎ complement.

Difference If the class CFL were closed under difference (and we know it is closed under $*$), then it would also be closed under complement:

$$\overline{L} = \Sigma^* - L.$$

Known CFL

$\blacksquare$ difference.