

Chapter 2.3 Non-Context-Free Languages.

Not all languages can be recognized by a PDA.

Consider $\{a^n b^n c^n \mid n \geq 0\} = "A^n B^n C^n"$

Imagine a PDA that recognizes $A^n B^n C^n$.

It might push every a onto the stack,
then pop & match the a 's with the b 's.

But then what?

That's not a proof that no such PDA can do the job,
but attempting to construct such a PDA will lead to
recognizing some limitations of PDAs.

To prove there is no such PDA, we need ...

Theorem 2.34 Pumping Lemma for CFLs.

\forall CFL A , \exists integer p such that,

$\forall s \in A$, $|s| \geq p$, $\exists u, v, x, y, z \in \Sigma^*$ where

$$s = uvxyz$$

where : 1. $uv^i x y^i z \in A$

2. $|v| \geq 1$

3. $|vxy| \leq p$

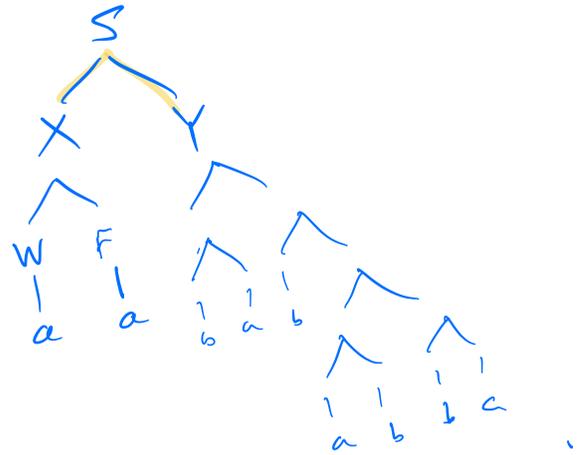
Proof idea: Let A be a CFL,

Let G be a CFG in **CNF** for A

Then all derivation trees in G are binary trees

(except at leaf).

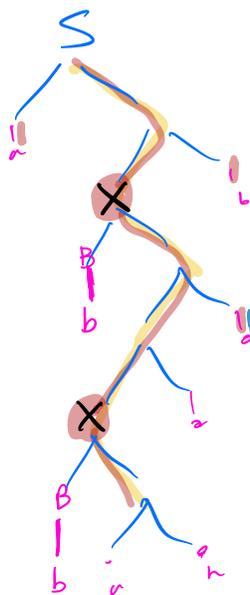
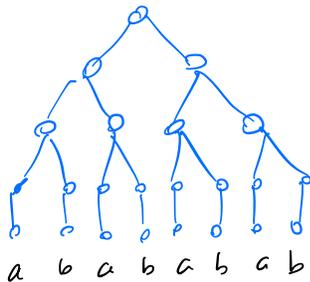
Then every string
of length $\geq P$
must have height \geq
 $\log_2 P + 1$.

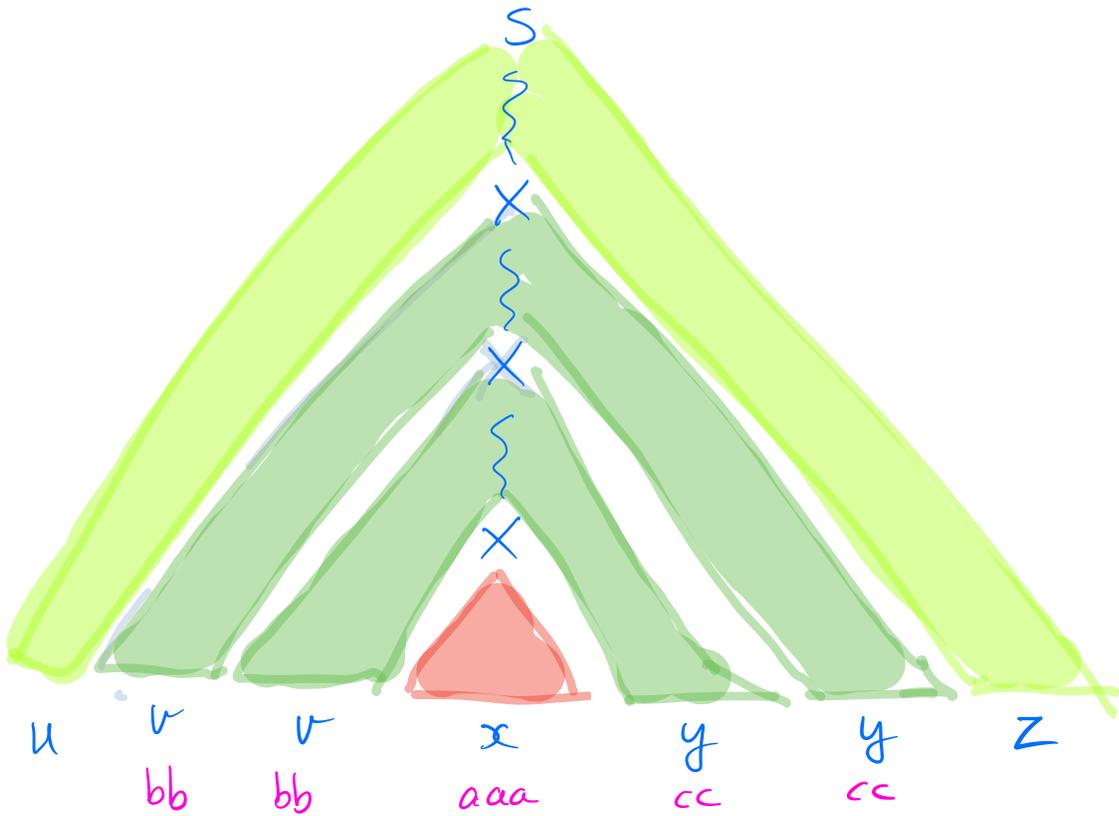
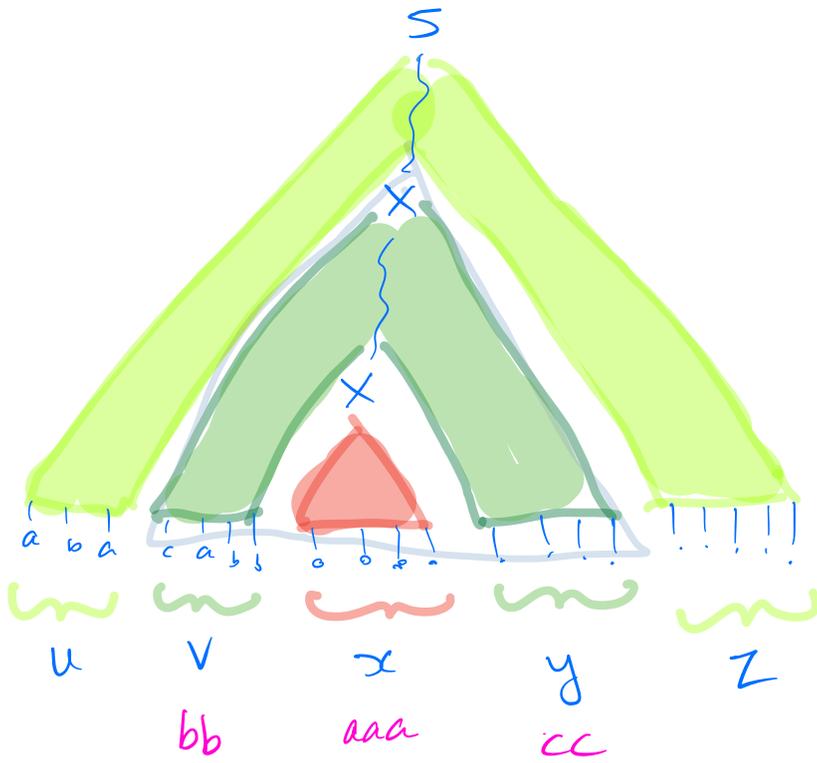


If $\log_2 P + 1 >$ number of variables in G .

then the longest path in the derivation must

reuse variables.





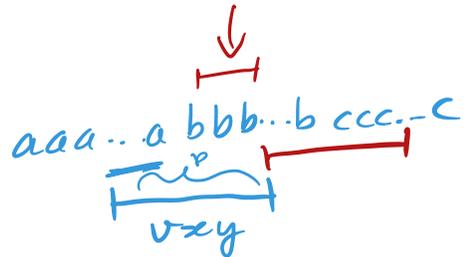
\exists a derivation for $uv^i x y^i z \quad \forall i = 0, 1, 2, \dots$

$$X \rightarrow \underbrace{aXYbaZ}_b \quad b\text{-ary.} \quad \square$$

Theorem: $A^n B^n C^n$ is not CF. $\{a^n b^n c^n \mid n \geq 0\}$

Proof: Suppose it were, and let p be its pumping constant.

Let $w = \underline{a^p b^p c^p}$.



$w \in A^n B^n C^n$ and $|w| \geq p$, so

$w = \underline{uv^1 x y^1 z}$ where $|vy^1| > 0$, $|vxy^1| \leq p$ ②.
 and $uv^i x y^i z \in A^n B^n C^n \quad \forall i \geq 0$, by P.L.

But ① and ② imply that v, x, y are such that:

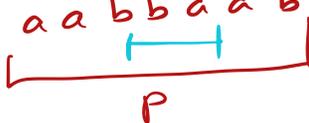
Case 1 - if v contains a 's then vxy does not contain c 's (because a 's are distance p from c 's)
 $\circ \circ$ pumping up yields more a 's than c 's. $\Rightarrow \Leftarrow$

Case 2 - if v does not contain a 's, then pumping up yields more b 's or more c 's than a 's. $\Rightarrow \Leftarrow$ □

Other languages that are not Context-Free:

$$\{ w\omega \mid w \in \{a,b\}^* \}$$

$$\{ w \in \{a,b,c\}^* \mid \#_a(w) = \#_b(w) = \#_c(w) \}$$

$$w = aabbaababab.$$


$$aabbbaabbbbaabc.$$


Claim: $L = \{w\#w \mid w \in \{a,b\}^*\}$ is not CF.

Proof: BWOC. \S L is CF and let p be its pumping constant.

Consider the string $a^p b^p \# a^p b^p$ which has length at least p so, by P.L.,

$$a^p b^p \# a^p b^p = uvxyz \text{ where}$$

$$|vxyz| \leq p \text{ and } uv^i x y^i z \in L \quad \forall i$$

case 1: $\#$ is in v or y .

\Rightarrow pumping down leads to a string with no $\#$.
 $\Rightarrow \Leftarrow$ (not in L).

case 2: $\#$ is in x

$\Rightarrow v$ is b^t for some t
and $y = a^r$ for some r .
 $r+t > 0$.

\Rightarrow pumping up: more b 's before $\#$ than after,
or: more a 's after $\#$ than before.

~~$\Rightarrow \Leftarrow$~~

case 3: # is not in vxy .

Pump up once.

\Rightarrow length of string before # is different
from the length of string after #.

\Rightarrow

$\therefore L$ is not CF. \square