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Pumping Lemma

– using it to prove languages are not regular –

For each of the following languages, either prove it is regular or prove it is not regular

$L_1 = \{a^i b^j \mid i,j \geq 0 \text{ and } i+j=5\}$ bbbbb + abbbb + aabbb + aaabb + aaaab + aaaa

$L_2 = \{a^i b^j \mid i,j \geq 0 \text{ and } i-j=5\}$

$L_3 = \{a^i b^j \mid i,j \geq 0 \text{ and } i-j \equiv 0 \bmod 5\}$

$L_4 = \{a^i b^j \mid 0 \leq i \leq j \leq 2000\}$

$L_5 = \{w \in \{a,b\}^* \mid w = w_1 w_2 \text{ where } |w_1| = |w_2| \text{ and } \#_a(w_1) = \#_a(w_2)\}$ Tricky!

$L_6 = \{w \in \{a-z\}^* \mid \text{every letter in } w \text{ appears at least twice}\}$

e.g. unprosperousness

$L_7 = \{w \in \{a,b\}^* \mid \#(w_a) \neq \#(w_b)\}$

hint: use a closure theorem ⇐ we'll cover it today in class

$L_8 = \{ a^i b^j c^k \mid k = i + j \}$ // and $i, j, k \geq 0$

PL: $\forall$ RL L $\exists$ a constant p such that $\forall w \in L$, $|w| \geq p$, we have: $w = xyz$ such that i) $|y| > 0$, ii) $|xy| \leq p$, and $xy^iz \in L$ $\forall$ $i = 0, 1, 2, \ldots$

aaabbccccc

Claim :

9. $\{ a^i b^j c^k \mid \text{if } i=1 \text{ then } j=k \}$ Tricky!

$L_7 = \{ w \in \{a,b\} \mid \#_a(w) \neq \#_b(w) \}$

Claim: L_7 is not regular.

Proof: BWOC. Suppose L_7 is regular.

Then so is $\overline{L_7} \cap L(a^*b^*)$

Observe that $\overline{L_7} \cap L(a^*b^*) = A^n B^n$

But $A^n B^n$ is not regular!

$\Rightarrow\Leftarrow$

$\therefore$ L_7 is not regular. $\blacksquare$

$L_5 = \{ w \in \{a,b\}^* \mid w = w_1 w_2,\ |w_1| = |w_2|, \text{ and}$
$\#_a(w_1) = \#_a(w_2) \}$

Claim: L_5 is not regular.

Proof:

Consider the string $b^p a a b^p$

That string is $\in L_5$ and has length $\geq p$, so by P.L., $b^p aab^p = xyz$ where $|y|>0$, $|xy| \le p$

and $xy^iz \in L_5$ $\forall\, i = 0, 1, 2, \ldots$

Note that the halfway mark is between the a's:

$$\underbrace{\overbrace{bb\cdots b}^{p}\,a}_{w_1}\;\Big|\;\underbrace{a\,bb\cdots b}_{w_2}.$$

and that $y = b^t$ for some $t > 0$.

Hence pumping up <u>once</u> will add t b's

into first block of b's:

- if t is odd, resulting string is $\notin L_5$ because it is of odd length.

- if t is even, result: halfway mark $\underbrace{bbb\cdots b}_{w_1}\;\Big|\;\underbrace{b\,aa\,bbbb\cdots b}_{w_2}$

and this string's first half has 0 a's, while the second half has 2 a's.

so resulting string is $\notin L_5$

$\Rightarrow\Leftarrow$

$\therefore$ L_5 is not regular. $\blacksquare$

Practice using closure Theorems

1. $L = \{ w \in \{a,b\}^* \mid \#_a(w) \neq \#_b(w) \}$

Claim: L is not regular.

Proof: BWOC. Suppose L is regular.

Then so is $\bar{L} \cap L(a^*b^*)$, by Closure Thms.

Note that $\bar{L} \cap L(a^*b^*) = A^n B^n$,

which we have shown is not regular.

$\Rightarrow\Leftarrow$

$\therefore$ L is not regular. ▨

2. $L = \{ w \in \{a,b\}^* \mid$ w has ≥ 3 b's, and the number of a's between first 2 b's is $\leq$ number of a's between 2^{nd} 2 b's$\}$

Claim: L is not regular.

Proof: BWOC. Suppose L is regular.

Then so is $L \cap L(ba^*ba^*b)$,

by the Closure Theorems for RLs.

Note that $L \cap L(ba^*ba^*b) = \{ ba^n ba^n b \mid n \geq 0 \}$.

Call that language L'.

Claim2: L' is not regular.

Proof: BWOC. Suppose L' is regular.

Let p be its pumping constant,

Consider the string $w = ba^pba^pb$.

Then by P.L., since $w \in L'$ and $|w| \geq p$,

$w = xyz$ where $|xy| \leq p$ ①, $|y| > 0$ ②,

and $xy^iz \in L'$ $\forall i \geq 0$.

By ① + ②, we have 2 cases for y:

case 1: $y = ba^t$ for some $t \geq 0$.

Pumping up once yields a string with too many b's to be in L'.

$\Rightarrow\Leftarrow$

case 2: $y = a^t$ for some $t \geq 1$

Pumping up once yields $ba^{p+t}ba^pb$, $t \geq 1$, which is not a string in L'.

$\Rightarrow\Leftarrow$

$\therefore$ L' is not regular. ▨ (Claim 2)

Since L' is not regular, neither is L. ▨