Warm up ... Jan 22, 2026

$\{ w \in \{0,1\}^* \mid w \text{ has both } 00 \text{ and } 11 \text{ as substrings} \}$

$$(0+1)^* 00 (0+1)^* 11 (0+1)^* + (0+1)^* 11 (0+1)^* 00 (0+1)^*$$

$\{ w \in \{a,b\}^* \mid w \text{ contains} \geq 2 \text{ b's that are not immediately followed by an } a \}$

$$(a+b)^* bbb (a+b)^* + (a+b)^* bb (a+b)^* bb (a+b)^* + (a+b)^* bb$$
$$+ (a+b)^* bb (a+b)^* b$$

$\emptyset$

$\emptyset$

Recall from last time...

Pumping Lemma:

$\forall$ Reg language L (over an alphabet Σ)

$\exists$ integer $p > 0$ such that

$\forall$ $w \in L$, $|w| \geq p$, we have that

$\exists$ $x, y, z \in \Sigma^*$ where $w = x \cdot y \cdot z$

and

1. $\forall$ $i \geq 0$ $xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$.

Note: Last time A^nB^n is not regular

Let $L_{eq} = \{ w \in \{a,b\}^* \mid \#_a(w) = \#_b(w) \}$.

Claim: L_{eq} is not regular.

Proof: BWOC. S'pose L_{eq} is regular. Then it has a pumping constant p.

Consider the string $a^p b^p$

$a^p b^p \in L$, and $|a^p b^p| \geq p$ so by P.L.:

$a^p b^p = xyz$ where $|y| > 0$, $|xy| \leq p$ and $xy^i z \in L$.

$$\overbrace{aaa\ldots a}^{p}\;\overbrace{bbb\ldots b}^{p}$$

$\leftarrow y \rightarrow$

Observe: $y = a^t$ for some $t > 0$.

$\Rightarrow$ pumping down once must yield another string in L_{eq}, by the P.L.

$\therefore\ a^{p-t} b^p \in L$, $t > 0$

$\Rightarrow\Leftarrow$ (contradicts defn of L_{eq})

$\therefore\ L_{eq}$ is not regular. $\blacksquare$

$$a^p b^p = \underbrace{aaaa\ldots a}_{p}\;\underbrace{bbbb\ldots b}_{p}$$

Balanced Strings of Parentheses

Let $\text{Bal} = \{ w \in \{ (,) \}^* \mid \text{the parens are balanced} \}$

e.g. $(())()$, $()$, $(())$, $(()())()(())$

But <u>NOT</u> $)$, $)($, $(()))(()$

aside: what makes a string of parens "balanced"?

- $\#_((w) = \#_)(w)$
- $\forall$ prefixes s of w, $\#_((s) \geq \#_)(s)$

Claim: Bal is not regular.

Proof: BWOC. Suppose Bal is regular. Then it has a pumping constant p.

Consider the string $(^p)^p$ $\quad \underbrace{((((\ldots(}_{p}\underbrace{)))))\ldots)}_{p}$

$(^p)^p \in \text{Bal}$, and $|(^p)^p| \geq p$.

$\therefore$ by P.L., $(^p)^p = xyz$ where $y \neq \varepsilon$

and $|xy| \leq p$ and $xy^iz \in L \ \forall i \geq 0$.

$\Rightarrow y = (^t$ for some $t > 0$.

$\Rightarrow xy^2z \in Bal$ ie $(^{p+t})^p \in Bal$ by P.L.

$\Rightarrow\Leftarrow$ it is not balanced.

$\therefore$ Bal is not regular $\blacksquare$

Let $EvenPal = \{ww^R \mid w \in \{a,b\}^*\}$

Claim: EvenPal is not regular.

Proof: BWOC. Suppose EvenPal is regular.

Then it has a pumping constant p.

Consider the string $a^p bb a^p$

$a^p bba^p \in EvenPal$ and $|a^p bba^p| \geq p$; hence

$a^p bba^p = xyz$ s.t. $|y| > 0$, $|xy| \leq p$, and

$xy^iz \in EvenPal \ \forall i \geq 0 \therefore y = a^t$ for some $t > 0$

$\Rightarrow a^{p-t} bba^p \in EvenPal$

$\Rightarrow\Leftarrow$ $\therefore$ EvenPal is not regular $\blacksquare$

$\text{Prime} = \{ a^n \mid n \text{ is prime} \}$. aa, aaa, aaaaa, ...

Claim: Prime is not regular.

Proof: BWOC. ∮ Prime is regular.

Then it has a pumping constant, call it p.

Let q be smallest prime > p+1

$a^q \in \text{Prime}$, and $|a^q| \geq p$ so by P.L.:

$a^q = xyz$ where $|y| > 0$, $|xy| < p$ and $xy^iz \in \text{Prime}$ $\forall i \geq 0$.

i.e $a^{|x|} \cdot a^{i \cdot |y|} \cdot a^{|z|} \in \text{Prime}$ $\forall i \geq 0$.

Note that $|x| + |z| \geq 2$. [How do we know this?]

Take $i = |x| + |z|$.

By P.L., $a^{|x|} \cdot a^{(|x|+|z|) \times |y|} a^{|z|} \in \text{Prime}$.

ie $|x| + |z| + (|x|+|z|) \times |y|$ is prime number.

But it has a factor $|x| + |y|$ which is ≥ 2!

$\Rightarrow \Leftarrow$ ∴ Prime is not regular. ▨

Closure Theorems for Regular Languages

RL is closed under $\cap$, $\cup$, complement, reverse.

Use closure Theorems to get simpler proofs of non-regularity. A^nB^n is not reg.

Claim: L_{eq} is not regular.

Proof: BWOC. S L_{eq} is regular.

Then so is $L_{eq} \cap L(a^*b^*)$

i.e. so is A^nB^n

$\Rightarrow\Leftarrow$ (we already proved A^nB^n is not reg)

$\therefore$ L_{eq} is not regular. $\blacksquare$

We can use closure Theorems...

given a difficult L, Suppose it is reg.

Then so is $L \cap$ some known reg Lang

and $\overline{L}$

and $L \cup$ some known reg lang

and one of these is better behaved, easy to use P.L. on, or is already known to be Non-regular.