

Warm up ...

Jan 22, 2026

$\{w \in \{0,1\}^* \mid w \text{ has both } 00 \text{ and } 11 \text{ as substrings}\}$

$$(0+1)^* 00 (0+1)^* 11 (0+1)^* + (0+1)^* 11 (0+1)^* 00 (0+1)^*$$

$\{w \in \{a,b\}^* \mid w \text{ contains } \geq 2 \text{ b's that are not immediately followed by an a}\}$

$$(a+b)^* b b b (a+b)^* + (a+b)^* b b (a+b)^* b b (a+b)^* + (a+b)^* b b$$
$$+ (a+b)^* b b (a+b)^* b$$

Recall from last time...



Pumping Lemma:

$\forall$  Reg language  $L$  (over an alphabet  $\Sigma$ )

$\exists$  integer  $p > 0$  such that

$\forall w \in L, |w| \geq p$ , we have that

$\exists x, y, z \in \Sigma^*$  where  $w = x y z$

and

1.  $\forall i \geq 0 \quad x y^i z \in L$

2.  $|y| > 0$

3.  $|x y| \leq p$ .

Note: Last time  
 $A^n B^n$  is not regular

Let  $L_{eq} = \{ \omega \in \{a,b\}^* \mid \#_a(\omega) = \#_b(\omega) \}$ .

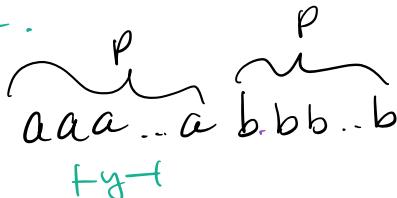
Claim:  $L_{eq}$  is not regular.

Proof: BWOC. If  $L_{eq}$  is regular. Then it has a pumping constant  $P$ .

Consider the string  $a^P b^P$   
 $a^P b^P \in L$ , and  $|a^P b^P| \geq P$  so by P.L.:

$a^P b^P = xyz$  where  $|y| > 0$ ,  $|xy| \leq P$  and

$xy^i z \in L$ .



Observe:

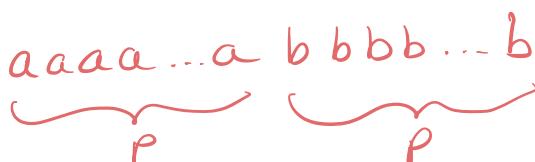
$y = a^t$  for some  $t > 0$ .

$\Rightarrow$  pumping down once must yield another string in  $L_{eq}$ , by the P.L.

$\therefore a^{P-t} b^P \in L$ ,  $t > 0$

$\Rightarrow$  (contradicts defn of  $L_{eq}$ )

$\therefore L_{eq}$  is not regular.  $\square$

$a^P b^P =$  

## Balanced Strings of Parentheses.

Let  $\text{Bal} = \{ w \in \{(),)\}^* \mid \text{the parens are balanced} \}$

e.g.  $((())()$ ,  $(()$ ,  $((())$ ,  $((()())())()$

But NOT  $)$ ,  $)()$ ,  $((())())()$

aside: what makes a string of parens

"balanced"?

$$\bullet \#_c(w) = \#_s(w)$$

$$\bullet \forall \text{ prefixes } s \text{ of } w, \#_c(s) \geq \#_s(s)$$

Claim:  $\text{Bal}$  is not regular.

Proof: Bwoc.  $\nsubseteq$   $\text{Bal}$  is regular. Then it has

a pumping constant  $P$ .

Consider the string  $(^P)^P$  

$(^P)^P \in \text{Bal}$ , and  $|(^P)^P| \geq P$ .

∴ by P.L.,  $(^p)^p = xyz$  where  $y \neq \epsilon$

and  $|xyz| \leq p$  and  $xy^iz \in L \forall i \geq 0$ .

$\Rightarrow y = a^t$  for some  $t > 0$ .

$\Rightarrow \underline{xyz} \in \text{Bal}$  ie  $(^{p+t})^p \in \text{Bal}$  by P.L.

$\Rightarrow$  it is not balanced.

∴ Bal is not regular  $\square$

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Let EvenPal =  $\{ww^R \mid w \in \{ab\}^*\}$

Claim: EvenPal is not regular.

Proof: By contradiction. Suppose EvenPal is regular.

Then it has a pumping constant  $p$ .

Consider the string  $a^p b b a^p$

$a^p b b a^p \in \text{EvenPal}$  and  $|a^p b b a^p| \geq p$ ; hence

$a^p b b a^p = xyz$  s.t.  $|y| > 0$ ,  $|xyz| \leq p$ , and

$xy^iz \in \text{EvenPal} \forall i \geq 0$ . ∴  $y = a^t$  for some  $t > 0$

$\Rightarrow a^{p-t} b b a^p \in \text{EvenPal}$

$\Rightarrow$  ∴ EvenPal is not regular  $\square$

Prime =  $\{a^n \mid n \text{ is prime}\}$ . aa, aaa, aaaaa, ...

Claim: Prime is not regular.

Proof: BWOC.  $\nsubseteq$  Prime is regular.

Then it has a pumping constant, call it  $p$ .

Let  $q$  be smallest prime  $> p+1$

$a^q \in \text{Prime}$ , and  $|a^q| \geq p$  so by P.L.:

$a^q = xyz$  where  $|y| > 0$ ,  $|xy| \leq p$  and  $xy^iz \in \text{Prime}$   
 $\forall i \geq 0$ .

i.e  $a^{|x|} \cdot a^i \cdot a^{|z|} \in \text{Prime} \quad \forall i \geq 0$ .

Note that  $|x| + |z| \geq 2$ . [How do we know this?]

Take  $i = |x| + |z|$ .

By P.L.,  $a^{|x|} \cdot a^{|z|} \cdot a^{(|x|+|z|) \times |y|} \cdot a^{|z|} \in \text{Prime}$ .

i.e  $|x| + |z| + (|x| + |z|) \times |y|$  is prime number.

But it has a factor  $|x| + |y|$  which is  $\geq 2$ !

$\Rightarrow \nsubseteq$   $\therefore$  Prime is not regular. 

## Closure Theorems for Regular Languages

RL is closed under  $\cap$ ,  $\cup$ , complement, reverse.

Use closure theorems to get simpler proofs of non-regularity.  $A^n B^n$  is not reg.

Claim:  $L_{eq}$  is not regular.

Proof: BWOC.  $\$ L_{eq}$  is regular.

Then so is  $L_{eq} \cap L(a^* b^*)$

i.e. so is  $A^n B^n$

$\Rightarrow \Leftarrow$  (we already proved  $A^n B^n$  is not reg)

$\therefore L_{eq}$  is not regular.  $\square$

We can use closure theorems...

given a difficult  $L$ , Suppose it is reg.

Then so is  $L \cap$  some known reg lang

and  $\bar{L}$

and  $L \cup$  some known reg lang

And one of these  $\lambda$  is better behaved, easy to use P.L. on, or is already known to be Non-regular.