Jan 22, 2026

Regular Languages

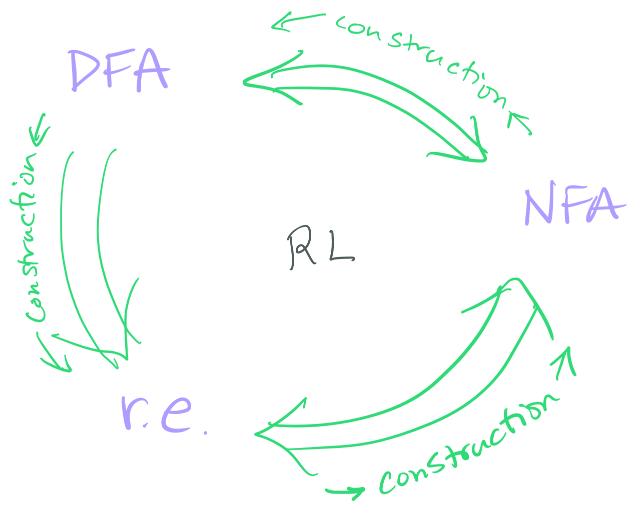
Defn: $L_1 \in RL \iff \exists$ a DFA recognizing L_1.

Thm: $\forall$ NFA $\exists$ an equivalent DFA

(Also: all DFAs are also NFAs)

Thm: $\forall L_1$:

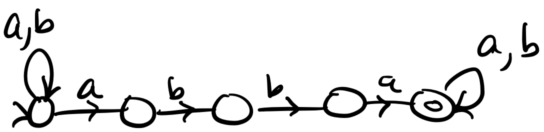
$\exists$ r.e. r s.t. $L(r) = L_1$ $\iff$ $\exists$ a NFA M s.t. $L(M) = L_1$



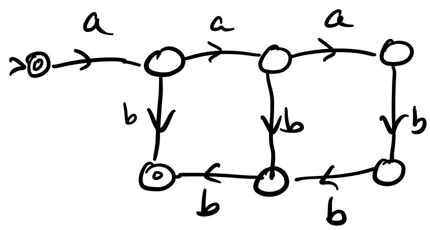
We will do more examples of using these constructions in tutorial.

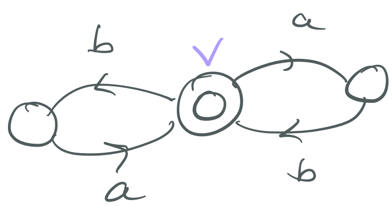
Examples of some RLs:

$(a+b)^* abba (a+b)^*$



$\{ a^n b^n \mid \forall n \in W \}$



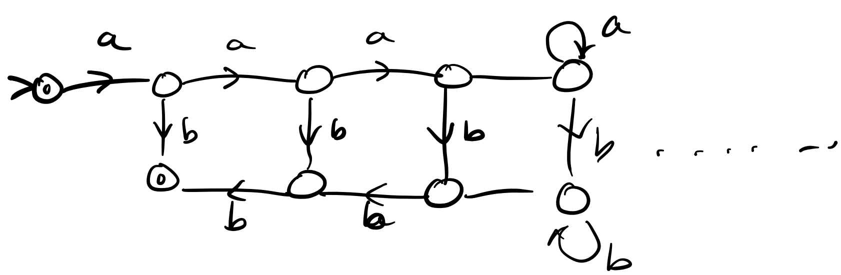


$\{ w \in \{a,b\}^* \mid \#_a(w) = \#_b(w) \leq 2 \}$

Homework: 2.5, 2.6, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18.2, 2.20

Are all languages Regular?

How about $\{ a^n b^n : n \geq 0 \}$?



That' a scheme, but is not a FA

How do we show that such a FA cannot exist?

BWOC – "By way of contradiction".

Intuition: If the number of states is <u>finite</u>, the string will have to "reuse" states it has already visited... $\to q \to$

But that means, in effect, that all ways of getting to $\to q \to$ are "equivalent", w.r.t. L.

2.9 Non-regular Languages

Pumping Lemma:

$\forall\ L \in RL \quad \exists$ pos int p such that

$\forall\ w \in L$ where $|w| \geq p$

$\exists$ strings x, y, z where

$w = xyz$ and

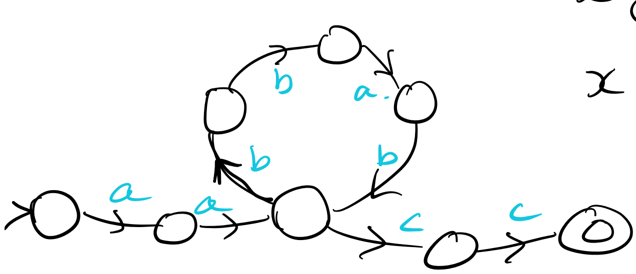
1. $\forall\ i \geq 0 \quad xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$.

Proof idea:

$w = xy^1z = aabbabcc$

$xy^0z = aa\ cc$

$xy^2z = aabbabbbabcc$



Proof of Pumping Lemma:

Let $w = \sigma_1 \; \sigma_2 \; \sigma_3 \; \cdots \; \sigma_n$

$$\begin{array}{ccccccccccc} & \sigma_1 & & \sigma_2 & & \sigma_3 & & \cdots & & \sigma_n & \\ q_1 & & q_2 & & q_3 & & & & q_n & & q_{n+1} \end{array}$$

be a string accepted by a DFA M that accepts L(M), and let the number of states in M be p, and $|w| = n \geq p$.

Then the "state visits" are $q_1, q_2, \cdots, q_{n+1}$

— i.e. $\exists$ more state visits than states, so some state is revisited.

$\$$ q_i and q_j are the same state, and furthermore they are the first repeated states in the sequence.

$\Rightarrow$ both occurrences appear within the first p symbols of w.

Then consider the substring y that "drives" M from q_i to q_j.

y is a substring of ω

ie $\omega = xyz$ for some x, y, z; strings

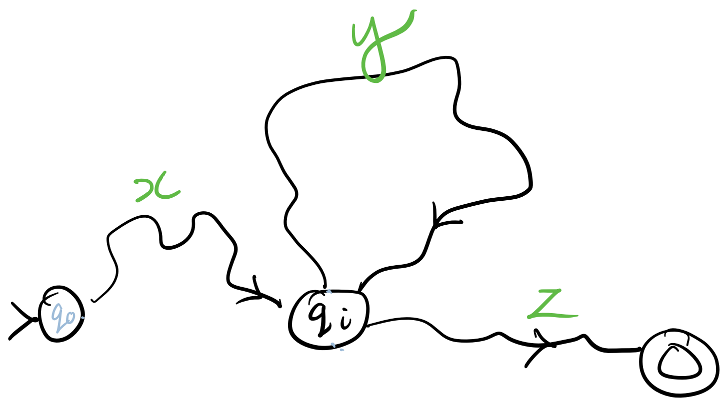
What would M do (accept or reject) on input

$= xyz$

$xyyz$

$x\,yyy\,z$

xz



That is, for a given reg. Lang L, there is a size limit (p) such that all strings $w \in L$ with length at least p have a non-empty substring y that appears entirely within the first p symbols of ω such that

y can be replaced with ε, yy, yyy, ...

and the resulting string is also in L. $\boxtimes$

for $\sigma \in \Sigma$ $\sigma^t = \underbrace{\sigma\sigma\sigma\ldots\sigma}_{t \text{ times}}$

for string y, we let

$$y^i = \underbrace{y \cdot y \cdot y \cdots y}_{i \text{ times.}}$$

$(ab)^5 = ababababab$

How does that help us?

$A^n B^n = \{ a^n b^n \mid n \geq 0 \}$

Theorem: $A^n B^n$ is not regular.

Proof: BWOC. Suppose $A^n B^n$ is regular.

$\Rightarrow \exists$ a pumping constant p such that all $w \in A^n B^n$, $|w| \geq p$ have a "pumpable" non-empty substring within the first p symbols, by Pumping Lemma.

which, when replaced by multiple copies of itself, yields another accepted string.

Consider the string $a^p b^p$.

This string has length $\geq p$, so P.L. applies.

$\Rightarrow \exists$ a substring within the first p symbols that is "y" (non-empty, and all pumping stays within A^nB^n)

$\Rightarrow$ then "y" must be a non-zero bunch of a's.

i.e. $y = a^t$ for some $t > 0$.

$\Rightarrow$ pumping up or down always yields a string in A^nB^n

$$\underbrace{a\,a\cdots a\,\underbrace{aa\cdots a}_{t}\,aa}_{p}\,\underbrace{b\,b\,b\cdots b}_{p} \in A^nB^n$$

therefore $\Downarrow$ pump up once

$$\underbrace{a\,a\cdots a\,\underbrace{aa\cdots a}_{t}\,\underbrace{aa\cdots a}_{t}\,aa}\,\underbrace{b\,b\,b\cdots b}_{p} \in A^nB^n$$

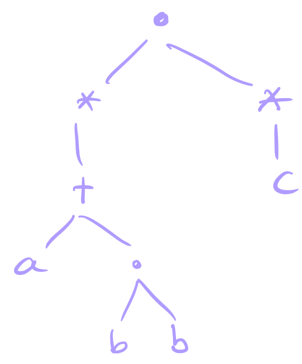
$\Rightarrow$ By P.L., $a^{p+t}b^{p} \in A^nB^n$, $t > 0$

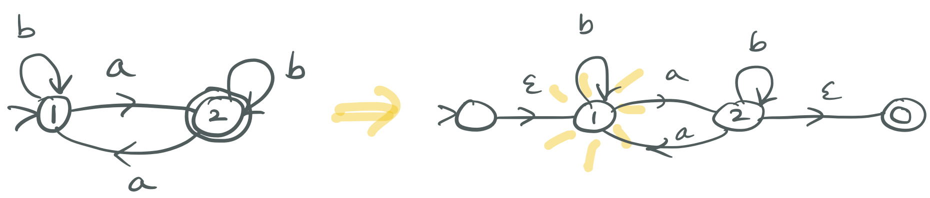
$\Rightarrow\Leftarrow$ (contradiction) $\Leftarrow$ what does it contradict?

$\Rightarrow$ Supposition was incorrect ... A^nB^n is not regular. ∎

Example uses of constructions (algorithms)

1. r.e $\Rightarrow$ FA. $(a+bb)^*\cdot c^*$



2. 

$A^n B^n$ is not regular.

Proof : BWOC. S $A^n B^n$ is regular and has pumping constant p.

Let $w = a^{\lceil p/2 \rceil} b^{\lceil p/2 \rceil}$

Then $\exists\, x, y, z$, $w = x \cdot y \cdot z$, and

1. $xy^i z \in L \;\forall\, i \geq 0$.
2. $|xy| \leq p$
3. $|y| > 0$., by P.L.

Then one of these cases applies (and each one leads to a contradiction):

i) $y = a^t$, $t > 0$ $\Longrightarrow$ did this case $\Rightarrow\!\Leftarrow$

ii) $y = a^t b^s$, $t > 0$, $s > 0$:

$\therefore$ by P.L., $xy^2 z \in A^n B^n$.

ie $aa \ldots a\, a^t b^s a^t b^s \ldots bb$

$\Rightarrow\!\Leftarrow$ (a's after b's is not possible in $A^n B^n$)

iii) $y = b^s$, $s > 0$:

$\therefore$ by PL, $xz \in A^n B^n$

i.e. $a^{\lceil p/2 \rceil} b^{\lceil p/2 \rceil - s} \in A^n B^n$.

$\Rightarrow\Leftarrow$

$\therefore$ $A^n B^n \notin RL$. $\blacksquare$