CSCI 260 2022 Assignment 1
Analysis of Algorithms, Master Theorem, Big-O
Submission Deadline: midnight, Saturday Oct 1, 2022.
Out of 38 marks

Handwritten is accepted in the original, not scanned. Electronically generated documents can be handed in electronically to gara.pruesse [at] viu.ca. 10% penalty for work submitted by Sunday at noon. 50% penalty for work submitted by Monday at noon. 100% penalty thereafter.

1. (1 marks) Explain briefly why the statement, "The running time of Algorithm A is at least $\mathbf{O}(n^2)$" is meaningless.

It's like saying "the running time is something more than something less than n^2". Meaningless.

2. (4 marks) Prove the "Reciprocal Rule": $f(n) \in \mathbf{O}g(n) \Rightarrow \frac{1}{g(n)} \in \mathbf{O}(\frac{1}{f(n)})$.

Suppose $f(n) \in O(g(n))$. Then $\exists$ c_1, n_1 positive constants such that $f(n) \leq c_1 \cdot g(n) \; \forall n \geq n_1$

$$\Rightarrow \frac{1}{c_1 \cdot g(n)} \leq \frac{1}{f(n)} \quad \forall n \geq n_1$$

$$\Rightarrow \frac{1}{g(n)} \leq c_1 \cdot \frac{1}{f(n)} \quad \forall n \geq n_1$$

$\therefore$ the same c_1, n_1 are also witnesses to the fact that $\frac{1}{g(n)} \in O(\frac{1}{f(n)})$

3. (4 marks) Show that $3n^2 + 4n - \frac{\log n}{\log \log n} \in \mathbf{O}(n^2)$ using the Definition of Big-O.

$$3n^2 + 4n - \frac{\log n}{\log \log n} \leq 3n^2 + 4n \quad \forall n \geq 11$$

since in this range, $\frac{\log n}{\log \log n} \geq 0$

$$\leq 3n^2 + 4n^2 \quad \forall n \geq 11$$

$$\leq 7n^2 \quad \forall n \geq 11$$

$\therefore$ $c = 7$, $n_0 = 10$ witness that

$3n^2 + 4n - \frac{\log n}{\log \log n} \in O(n^2)$

4. (4 marks) Prove that $6n \lg n + 4n^3 - 8n \in \Theta(n^3)$. You can use the Rules or the definition of Big Oh.

A.
1. $\lg n \in O(n)$ – less-than rule, $n_0 = 2$
2. $6n \in O(n^2)$ – Polynomial Rule
3. $6n \lg n \in O(n^3)$ – 1,2 Product Rule
4. $4n^3 - 8n \in O(n^3)$ – Polynomial Rule
5. $6n \lg n + 4n^3 - 8n \in O(n^3)$ –3,4 Sum Rule.

B.
1. $n^3 \in O(4n^3 - 8n)$ Polynomial Rule
2. $4n^3 - 8n \in O(6n \lg n + 4n^3 - 8n)$ Less than Rule
3. $n^3 \in O(6n \lg n + 4n^3 - 8n)$ 1,2 Transitivity

Since $n^3 \in O(6n \lg n + 4n^3 - 8)$, then $6n \lg n + 4n^3 - 8n \in \Omega(n^3)$

And from Part A, we know $6n \lg n + 4n^3 - 8n \in O(n^3)$

$\therefore$ $6n \lg n + 4n^3 - 8n \in \Theta(n^3)$

5. (6 marks) Is $\frac{n}{\log n} \in \mathbf{O}(n)$? In $\Omega(n)$? In $\Theta(n)$? Answer Yes or No to each, and prove your answers using the Facts or the definitions of Big Oh, Ω, and Θ.

We show $\frac{n}{\log n} \in O(n)$, and $\frac{n}{\log n} \notin \Omega(n)$, hence by definition $\frac{n}{\log n} \notin \Theta(n)$.

I.
1. $1 \in O(\log n)$ by Less Than Rule
2. $\frac{1}{\log n} \in O(1)$ 1, Recip. Rule
3. $n \in O(n)$ Constant Factors
4. $\frac{n}{\log n} \in O(n)$ 2,3 Product Rule.

II. Suppose $\frac{n}{\log n} \in \Omega(n)$

$\therefore$ $n \in O(\frac{n}{\log n})$.

$\therefore$ $\exists c, n_0$ s.t. $n \leq c \cdot \frac{n}{\log n}$ $\forall n \geq n_0$

$\therefore$ $\log n \leq c$ $\forall n \geq n_0$

$\therefore$ $n \leq 2^c$ $\forall n \geq n_0$

$\Rightarrow\Leftarrow$

$\therefore$ $\frac{n}{\log n} \notin \Omega(n)$.

6. (4 marks) List the following functions in a column so that the function at the top grows the slowest, and the function at the bottom grows the fastest. If two functions grow at the same rate asymptotically, put them on the same row.

n^n $\quad n!$ $\quad n + 10\sqrt{n}$ $\quad (2.3)^n$ $\quad n^2$ $\quad 2^{n \lg n}$ $\quad 10n \lg \lg n$
2^n $\quad n^2 \lg n$

$n + 10\sqrt{n}$

$10n \lg \lg n$

$n^2 \lg n$

2^n

$(2.3)^n \; n^2$

$n!$

$n^n, \; 2^{n \lg n}$ $\leftarrow$ $2^{n \lg n} = \left(2^{\lg n}\right)^n = n^n$

7. Briefly describe what happens to the running time of an algorithm with run time $f(n)$ when the size of the input is increased by one, and when the size of the input doubles:

   (a) (2 marks) $f(n) = n$

   Running time effect:

   +1: increases by a constant amount, regardless of n

   ×2: doubles

   (b) (2 marks) $f(n) = n^2$

   +1: increase by an amount that is approximately linear in n

   ×2: quadruples

   (c) (2 marks) $f(n) = 2^n$

   +1: doubles

   ×2: squares.

8. Solve the following Divide-and-Conquer recurrences using the Master Theorem.

(a) (3 marks) $T(n) = \underline{3T(n/4)} + n$ $\quad n^{\log_b a} = n^{\log_4 3} \simeq n^{0.8}$

$n \in \Omega(n^{0.8+\varepsilon})$

regularity check:

$3\left(\frac{n}{4}\right) < cn$ when $c = .76 < 1.$

$\therefore T(n) \in \Theta(n)$. by case 3 of MT.

(b) (3 marks) $T(n) = 4T(n/2) + \underbrace{n^2}_{f(n)}$ $\quad n^{\log_b a} = n^{\log_2 4} = n^2$

$f(n) = n^2 \in \Theta(n^{\log_4 2})$

$\therefore T(n) \in \Theta(n^2 \lg n)$

(c) (3 marks) $T(n) = 6T(n/3) + n \log n$ $\quad n^{\log_b a} = n^{\log_3 6} \approx n^{1.65}$

$n \log n \in O(n^{\log_3 6 - \varepsilon})$

$\therefore T(n) \in \Theta(n^{\log_3 6})$