Topic: Data Structures for ADT $\boxed{\text{Disjoint Sets}}$

Alg Approach: Data Structuring

Sometimes the main innovation is in the way the data is structured. An example is disjoint set operations.

Data: Elements $x_1, x_2, \ldots x_n$ (could be integers)

Organization: maintained as <u>disjoint sets</u>

$$S = \{ S_1, S_2, \ldots, S_k \}$$

- A set is non-empty
- A set is represented by one of its members
- The collection is <u>dynamic</u> (changes over time)

Operations:

MakeSet (x) : // x is not already in a set
- makes a new set That just contains x
- x is representative.

Union (x, y)
- Set containing x and set containing y

are unioned (one set)
- representative?

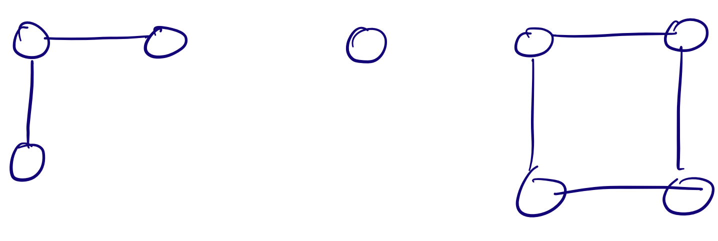
FindSet (x)
- returns a pointer (or name of) the representative of the unique set containing x

To analyze the running times of various implementations, we use two parameters:

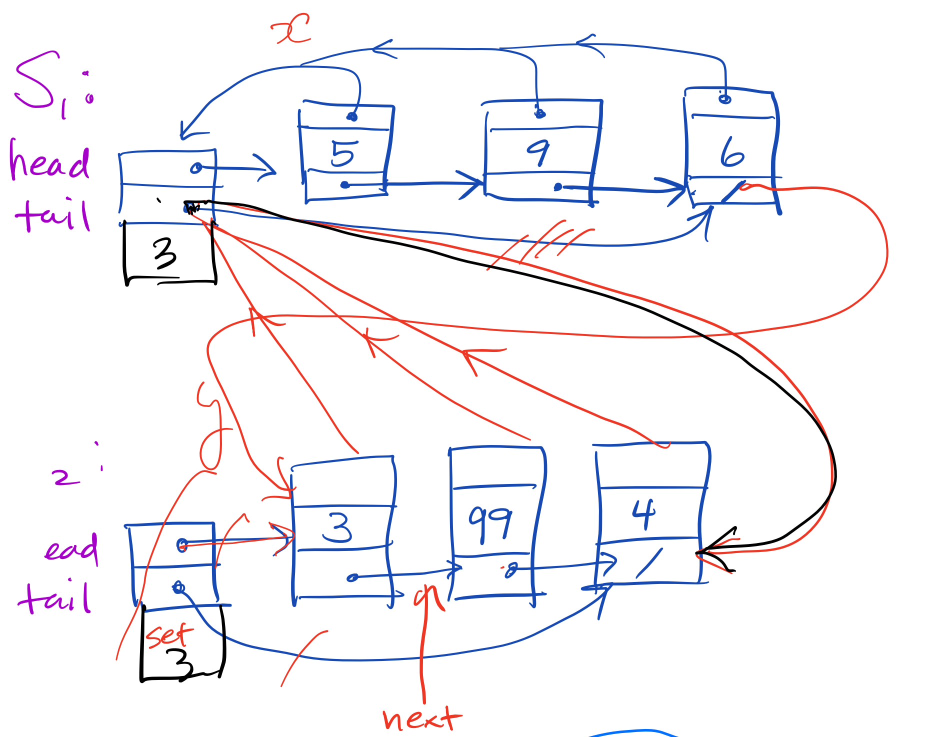
n = number of MakeSet operations

m = number of MakeSet, Union, FindSet operations.

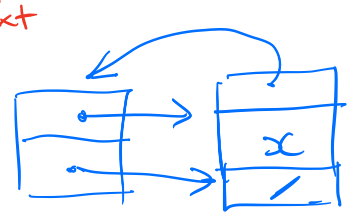
Eg Application.
Maintaining a collection of graph components (such as LANs)



Representing Disjoint Sets as Linked Lists



MakeSet (x) :



O(1)

FindSet (x) : return x → set → head

O(1).

Union (x, y) : ?

```
if ( FindSet(x) ≠ Find set(y) )
while temp ≠ NULL {
temp = x → set → head
temp → set = y → set
temp = temp → next
}
```

O(| y's set |)

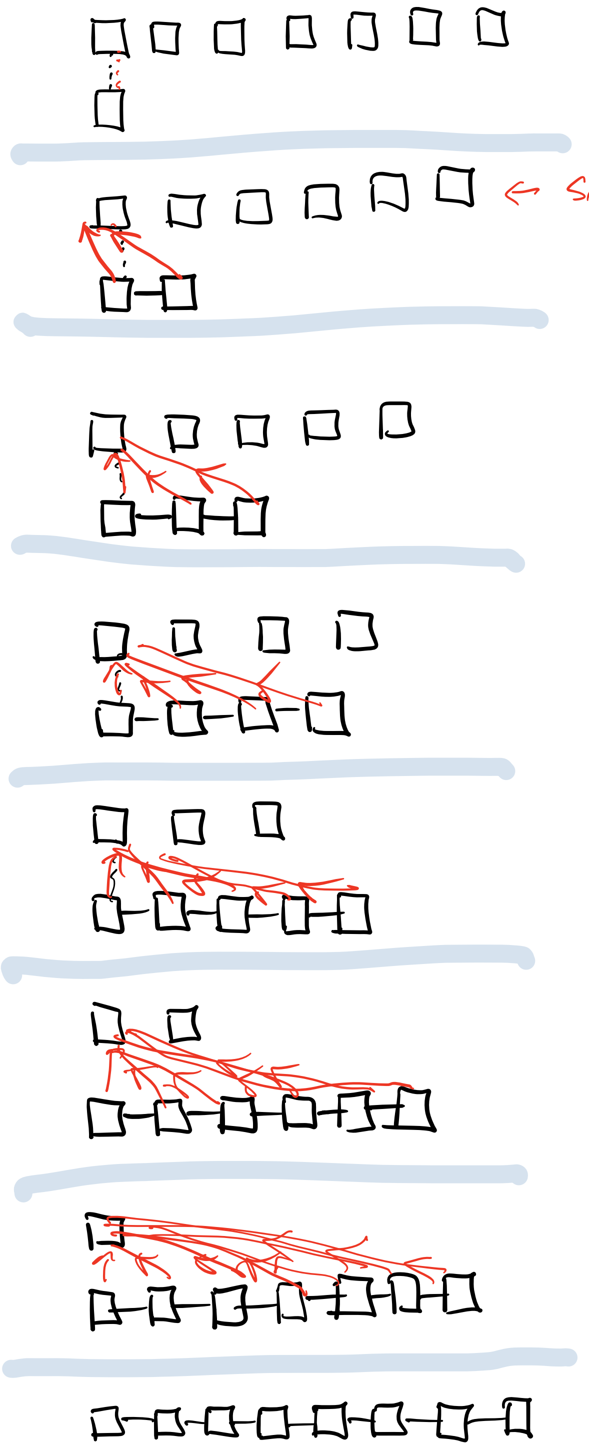
Ideas

First, what if we do no optimization.

Worst case # links

to reassign.

$O(1)$



← Singleton

2

3

4

5

6

7

$$\frac{(n-1)(n-2)}{2}$$

$$O(n^2).$$

What is total worst-case running time for m ops, n of which are MakeSet? $O(n^2)$ for m ops

in worst case, $m \in O(n)$ so amortized over m ops

How can we do better?

is $O(n)$ per operation.

<u>heuristic</u>: when unioning, choose the smaller list to reassign "set" pointers to larger list.

Weighted-Union heuristic

Introduction to Algorithms)
Theorem 21.1 (Cormen, Leiserson, Rivest + Stein)

Linked-list rep'n + weighted-union heuristic for m ops, n makeSets, takes $O(m + n \lg n)$ time

Worst case is when $m \approx c \cdot n$
Then $O(m+n\lg n) \approx O(n \lg n)$ and amortizing over $c \cdot n$ op's, for a constant c, yields $O(\lg n)$ time/op

Proof: We prove an upper bound on the number of times x's parent pointer can be reset.

After all m ops, whats the largest set-size x can be in? n

For x to have its pointer reset, x must be in the smaller of the two sets being unioned....
So what happens to set-size of x's set each time it's pointer is updated?

doubles (or more).

How many times can that happen before the set size is n?

$\lg n$

Proof: The max number of union ops is $n-1$.

$\forall x$, x's pointer is updated only if the size of the set containing x doubles.

This occurs $\le \lceil \lg n \rceil$ times during the m ops, as max set size is n.

$\therefore$ pointer updates $\in O(n \lg n)$

$\therefore$ total running time for m ops is $O(m + n \lg n)$

$\therefore$ worst case amortized running time is when

$m \approx n$ and amortized running time (time per operation, on average) is $O(\lg n)$. $\blacksquare$

For dynamic data structures, that have a sequence of operations applied to them, we analyze the worst case running time for m operations and then amortize (average) that running time over the m ops (ie divide by m).

Result is called the amortized running time per operation.

This is different from "Average case analysis"

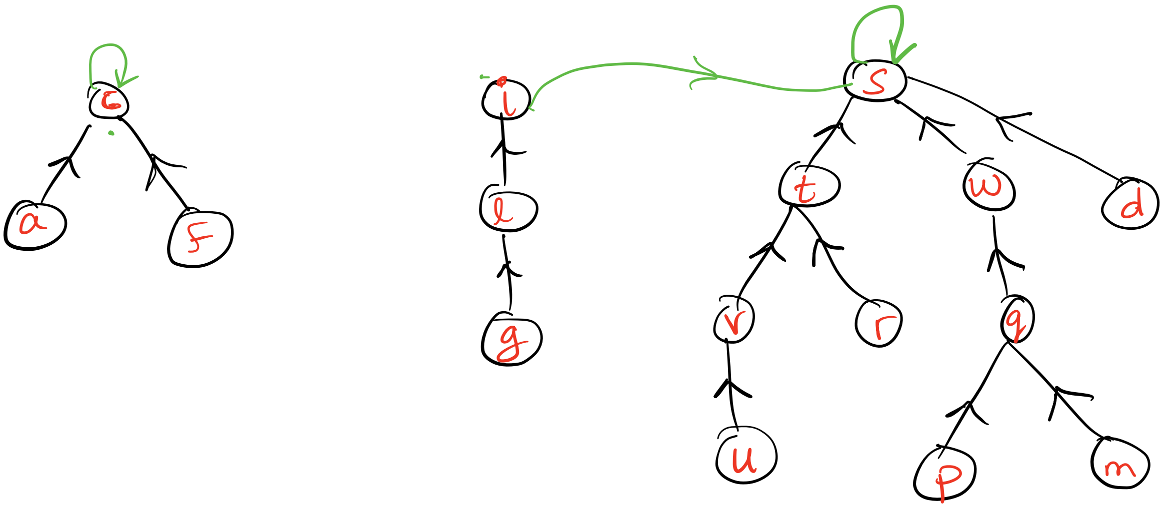
amortized = Worst case, amortized over # of ops.

"Average" running time = Expected running time

given an assumed distribution of the
input cases.

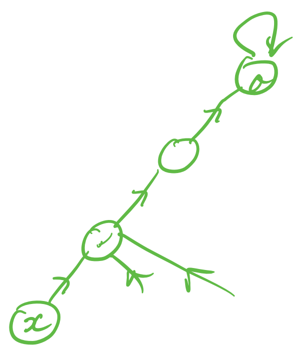
ADT Disjoint Sets

Represented by a Forest



MakeSet(x)

FindSet(x)



return a ptr to the root.

Union(x, y)