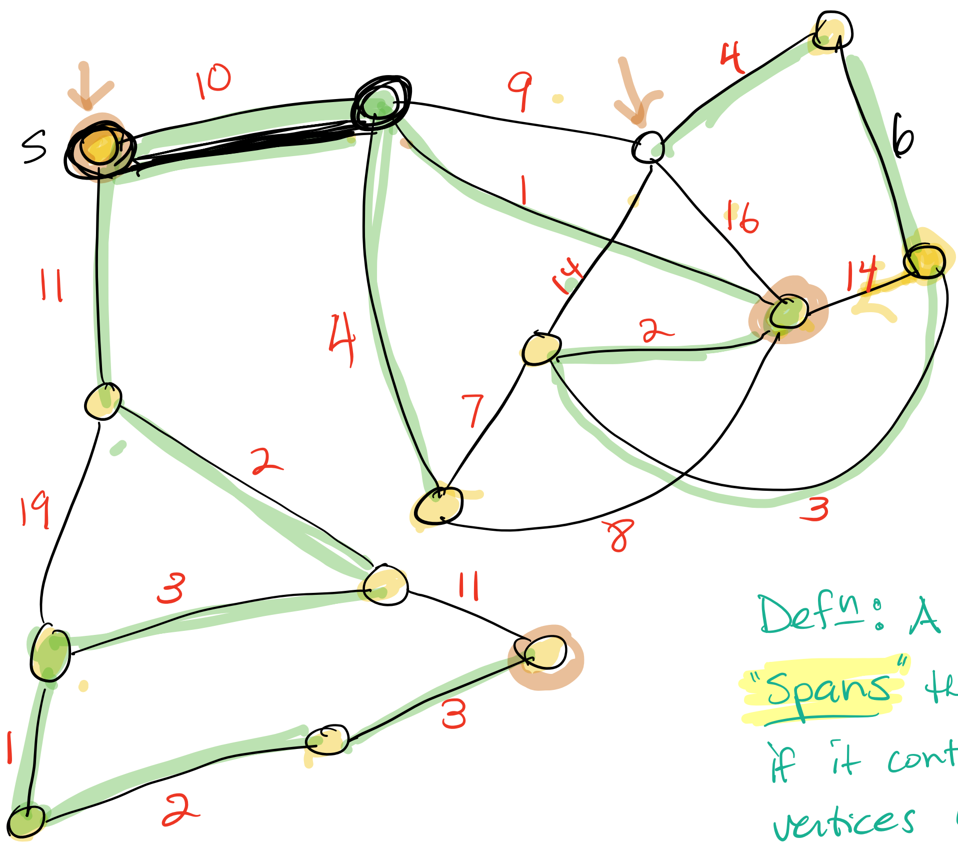
Algorithmic Approach #3: Greed

Let's consider weighted undirected graphs.
The weights here are on the edges.

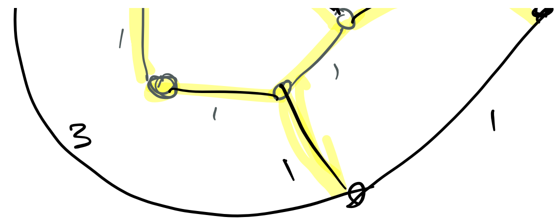


Defn: A subgraph "spans" the graph if it contains all the vertices of the graph.

Min Weight Spanning Tree (MST)

Input: Connected Graph $G=(V,E)$, weight function $w: E \to \mathbb{R}^+$

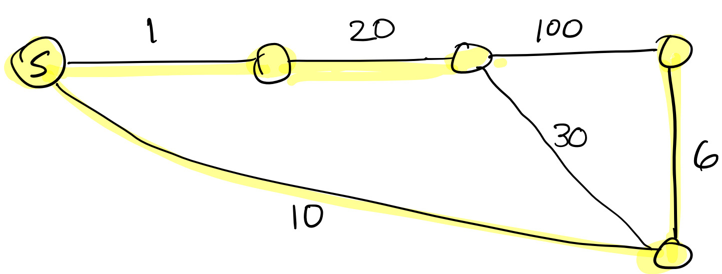
Postcondition: $T=(V,E')$, where $E' \subseteq E$ and
T is connected and T has $\min \sum_{e \in E'} w(e)$



Idea #1: Prim's

- Pick a start vertex v
- Grow a tree starting from v, always picking the lowest weight edge that can extend the tree.

from any node in current tree containing s.



Theorem: Prim's algorithm is CORRECT for MST... that is, it produces a tree of minimum weight.

Proof: First, we must argue that every connected graph has a spanning tree. How would you argue that? Left as an exercise for the student.

Given that we can prove that, we first define "promising"

Def^n: A tree T is promising if T is the subgraph of some MST T_{opt} of G.

Now let us consider a graph G, and Prim's algorithm applied to it.

Prim's will add edges in some order; lets call it

$e_1, e_2, e_3, \dots, e_{n-1}$

where $n = |V|$.

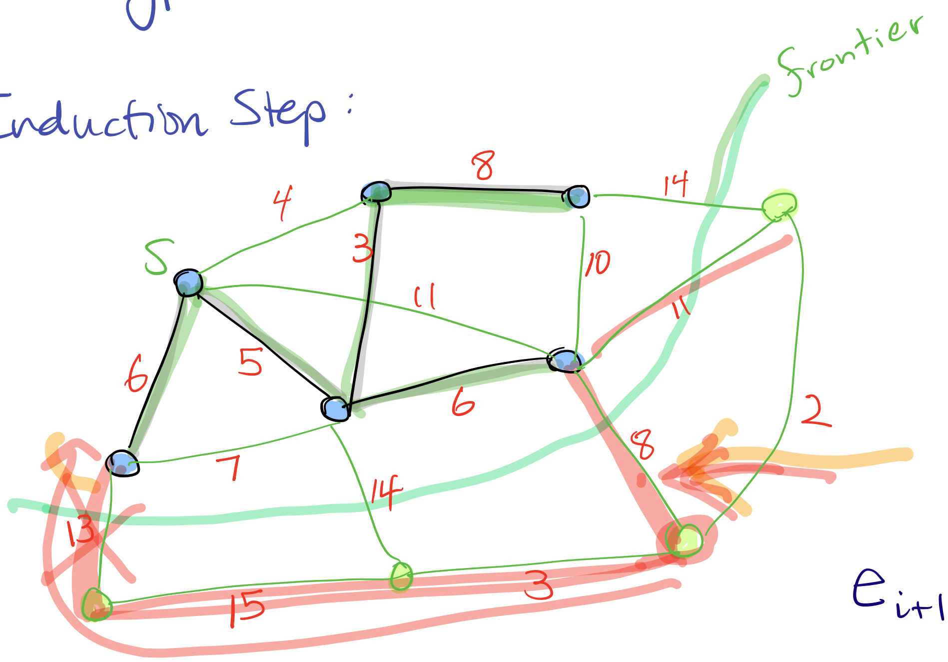
Claim: The tree $e_1 + e_2 + \dots + e_i$ is promising $\forall i,\ 0 \le i < n$.

Proof: By Induction on i.

Basis: $i = 0$. The subgraph with no edges is clearly promising.

Ind Hyp: $e_1 + e_2 + \dots + e_i$ is promising

Induction Step:



Note: in a tree, $\exists ! 1$ path from v to s $\forall v$.

Suppose ALL the MSTs that extend $e_1 + \dots + e_i$ do NOT contain e_{i+1} (which joins vertex v to the tree). Let T_{opt} be one MST.

Consider the v–s path in T_{opt} ...

it must cross the "frontier" using some edge e_j, $e_j \notin \{e_1, e_2, \dots, e_{i+1}\}$

and $w(e_j) \geq w(e_{i+1})$

Then $T_{opt} - e_j + e_{i+1}$ is also a spanning tree (think about it...)

... and it has weight no more than $w(T_{opt})$

$\therefore$ it is a MST,

and $\{e_1, e_2, \dots, e_i, e_{i+1}\}$ is promising. $\square$

Thus we have shown that all the edge sets $\{e_1, e_2, \dots, e_i\}$ are promising, for $0 \leq i < n$.

So $\{e_1, e_2, \dots, e_{n-1}\}$ is promising.

ie. T is promising, and T is a spanning tree.

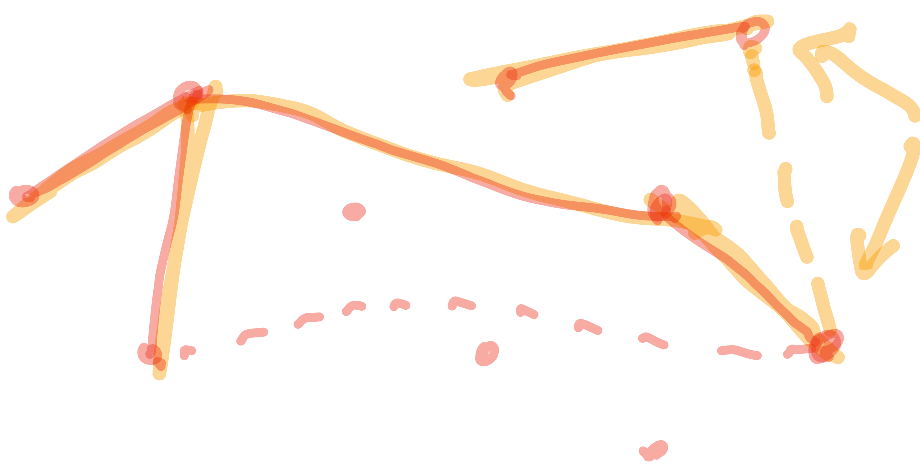
$\therefore$ T is a MST. $\blacksquare$

Kruskal's Algorithm for MST.

- sort edges by weight. $e_1\ e_2 \ldots e_m$

  least weight

- add the edges in the order of increasing weight, only add if don't form a cycle.



graph G1, G2;

G1. read_graph(   )

G2