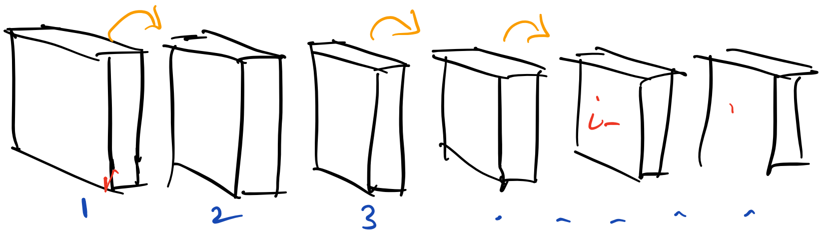
Induction – Review

Dominoes...



$\underbrace{P(i)}_{\text{Ind Hyp}} \rightarrow \underbrace{P(i+1)}_{\text{Prove this}}$

* I. If the n^{th} domino topples, so does the $n+1^{st}$

* II The first domino topples. ← Base.

If we know I and II then...

$\forall n$, we can construct a proof that all dominoes from 1 to n topple:

	Rationale
"1. d_1 topples	I
2. d_1 topples $\Rightarrow$ d_2 topples	– II, instantiation
3. d_2 topples	1, 2 Modus Ponens.
4. d_2 topples $\Rightarrow$ d_3 topples	– II, instantiation
5. d_3 topples	3, 4, MP.

⋮

2n-3. d_{n-1} topples

2n-2. d_{n-1} topples $\Rightarrow$ d_n topples

2n-2. d_n topples."

Having statements I + II

Induction Step (needs to be proved)

base case (needs to be proved)

P(1) or P(0)

property

$P(i) \to P(i+1)$.

Then we have an "algorithm" for writing a proof for $P(n)$ for any n, $n \geq$ base case!

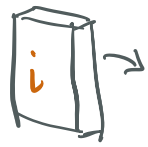
It must be a statement about integers

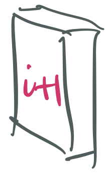
... but statements about graphs can be statements about integers...

Statement: All non-empty acyclic undirected graphs $T=(V,E)$ have $|E| = |V| - 1$

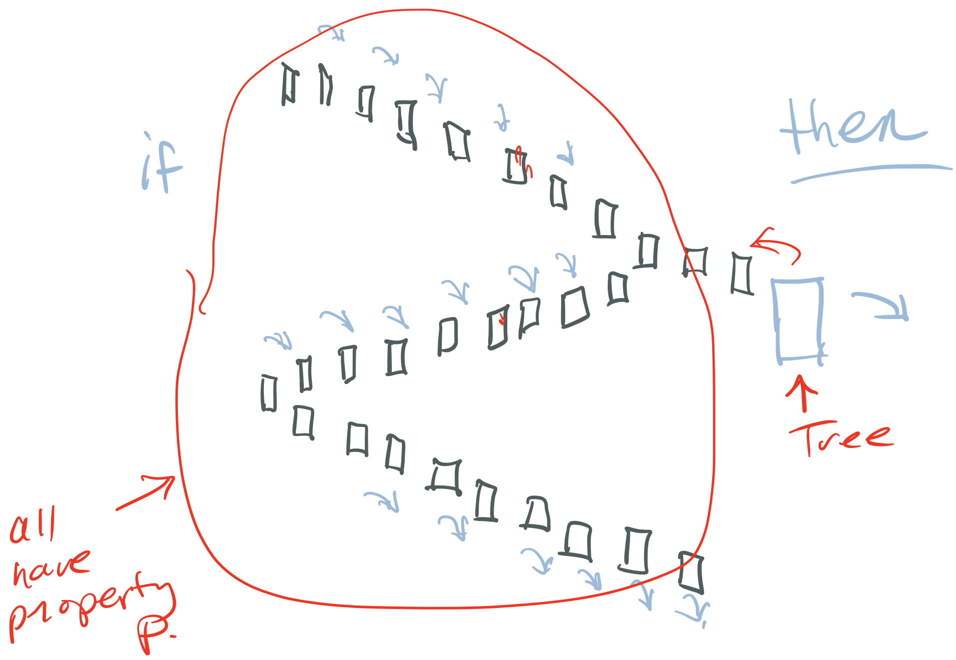
Statement: $\forall n, n \geq 1$, if $T=(V,E)$ is an acyclic undirected graph on n vertices, then $|E| = n-1$.

We would normally use Strong Induction for graphs.

Weak Induction: show if 

then 

Strong Induction:



Claim: $\forall\, n \geq 1$, if T is a tree on $|V|$ vertices, then T has $\underbrace{|V|-1}_{m}$ edges.

Proof:

Base case: $T = \bullet$ $\quad n=1$, $m=0$. ✓

(Fix n): Let n be an integer > 1.

Ind Hyp: Suppose

"all trees of $|V| < n$ have $|V|-1$ edges"

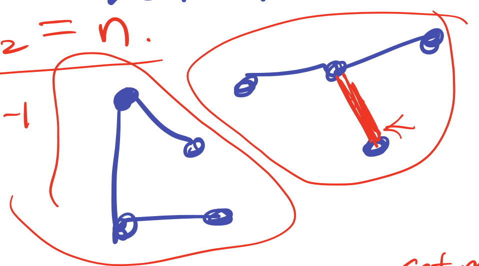
(weak induction: all trees of size $|V|-1$ have the property).

Ind Step:

Let T be any tree on $|V| = n$ vertices.

$n_1 + n_2 = n$.

$n_1 - 1 + n_2 - 1$ edges.



Let v be any vertex of degree 1 in T.

ie $n_1 + n_2 - 2$ edges.

set minus — remove vertex + edge.

Then $T \setminus v$ has $n-1$ vertices.

Adding back in we get

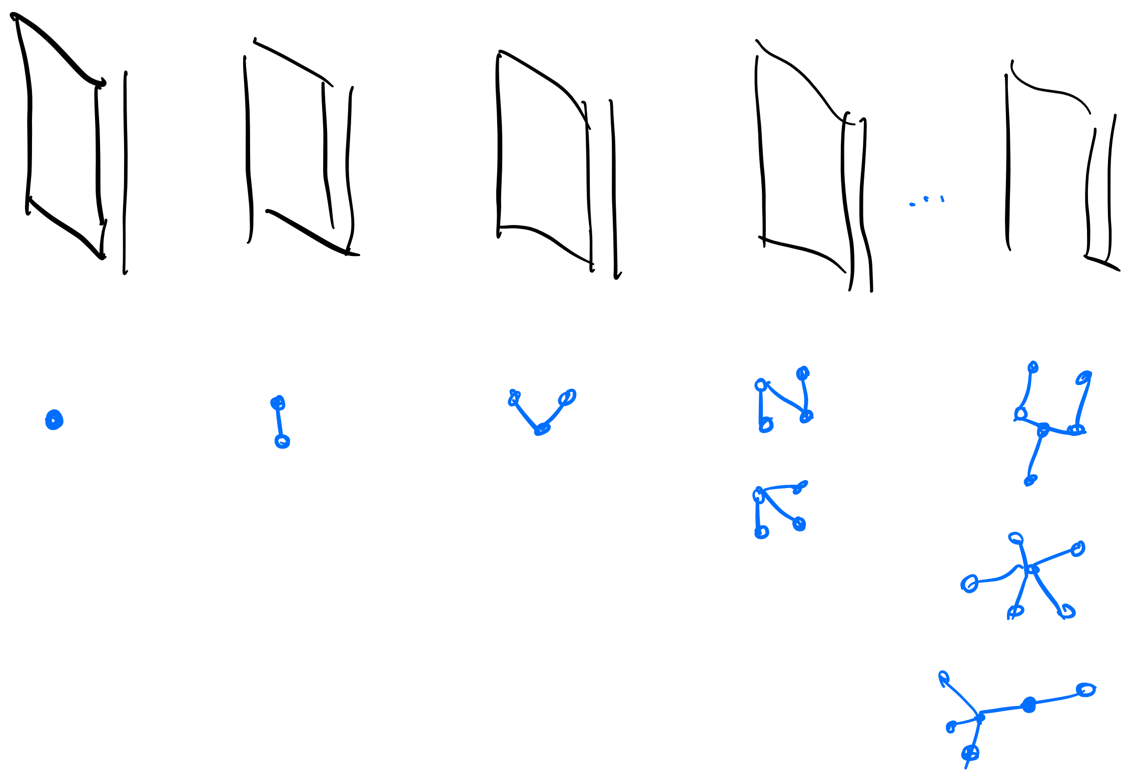
Then the Ind Hyp applies to $T \setminus v$

and $T \setminus v$ has $n-2$ edges. by Ind Hyp

$n_1 + n_2 - 1$ edges. ie $n-1$ edges.

$\Rightarrow$ T has $n-2+1$ edge (adding the one that connected v to rest)

ie T has n vertices and $n-1$ edges. $\blacksquare$



Claim: $\forall n \geq 1$, every tree with n nodes has $n-1$ edges.

Proof: Strong Induction on number of vertices in the tree.

Base: If $T = \bullet$, $n = 1$ and num edges $m = 0$. So claim is true for this case.

Ind Step: Let n' be a value > 1.

Ind Hyp: $\forall n < n'$, each tree of size n has $n-1$ edges.

Let T be any tree of size n'.

Since $n' > 1$, T has at least 2 vertices and at least 1 edge (since T is connected).

Let (u,v) be an edge in T.

Then $T \setminus \{(u,v)\}$ is the tree minus the edge, so is a forest of two trees T_1 + T_2, with number of vertices n_1 and n_2, respectively.

Note $n_1 + n_2 = n$, $1 \le n_1 < n'$, $1 \le n_2 < n'$

$\therefore$ T_1 has $n_1 - 1$ edges

T_2 has $n_2 - 1$ edges.

$\therefore$ number of edges in T

$$= \underbrace{(n_1 - 1)}_{\text{edges in } T_1} + \underbrace{(n_2 - 1)}_{\text{edges in } T_2} + \underbrace{1}_{(u,v)} = n' - 1$$

$\blacksquare$