# Final Prep: Complexity Analysis

Rules I won't ask you to prove:

SBTLD: Strange but true Log Domination Rule, ie $\log^r n \in O(n^s)$ $\forall r, \forall s > 0$

Log Base is Irrelevant:

$\log_a n \in O(\log_b n)$ $\forall a, b > 1$

Polynomial rule:

$f(n)$ a polynomial in $n$ of degree $K$

$g(n)$ a polynomial in $n$ of degree $K' \geq K$

$\Rightarrow f(n) \in O(g(n))$

Note: logarithms do not appear in polynomials!

## Let's prove the Reciprocal Rule

Suppose $f(n) \in O(g(n))$,

$\Rightarrow \exists\, c>0,\ n_0>0$ s.t. $f(n) \leq c \cdot g(n)\ \forall n \geq n_0$

$\Rightarrow \dfrac{1}{c \cdot g(n)} \leq \dfrac{1}{f(n)}\ \forall n \geq n_0$

$\Rightarrow \dfrac{1}{g(n)} \leq c \cdot \dfrac{1}{f(n)}\ \forall n \geq n_0$

$\Rightarrow \dfrac{1}{g(n)} \in O\left(\dfrac{1}{f(n)}\right)$, demo'd by same $c$, $n_0$ as for $f(n) \in O(g(n))$. $\blacksquare$

## Let's prove the Sum Rule

Suppose $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$

$\Rightarrow \exists\, c_1>0,\ n_1>0,\ c_2>0,\ n_2>0$ such that

$f_1(n) \leq c_1 \cdot g(n)\ \forall n \geq n_1$

$f_2(n) \leq c_2 \cdot g(n)\ \forall n \geq n_2$

$\Rightarrow f_1(n) + f_2(n) \leq c_1 \cdot g(n) + c_2 \cdot g(n)$
$\forall n \geq \max(n_1, n_2)$

$\Rightarrow f_1(n) + f_2(n) \leq (c_1+c_2)\, g(n)\ \forall n \geq \max(n_1, n_2)$

$\Rightarrow f_1(n) + f_2(n) \in O(g(n))$
emo'd by $c = (c_1+c_2)$, $n_0 = \max(n_1, n_2)$.

Do some Big-Oh proofs:

Claim: $3n^2+4n+6 \in O(n^2-n)$, using defn.

Proof: $3n^2+4n+6 \le 3n^2 + n^2 + n^2 \quad \forall n \ge 4$

$\le 5n^2 + n^2 - 6n \quad \forall n \ge 6$

$\le 6n^2 - 6n \quad \forall n \ge 6$

$\le 6(n^2-n) \quad \forall n \ge 6$

$\therefore 3n^2+4n+6 \in O(n^2-n)$ as demo'd by $c=6$, $n_0=6$.

Claim: $\log n \in O\left(\frac{n}{\log n}\right)$

Proof: Using the rules.

1. $\log^2 n \in O(n)$ SBTLD rule
2. $\frac{1}{\log n} \in O\left(\frac{1}{\log n}\right)$ CF
3. $\log n \in O\left(\frac{n}{\log n}\right)$ 1,2 Product Rule. $\blacksquare$

Claim: $2n \lg n \notin O(n)$

Proof: BWOC. Suppose $2n \lg n \in O(n)$

$\Rightarrow \exists\ c>0, n_0>0$ such that

$2n \lg n \leq c \cdot n \quad \forall n \geq n_0$

$\Rightarrow \lg n \leq \frac{c}{2} \quad \forall n \geq n_0$

$\Rightarrow 2^{\lg n} \leq 2^{c/2} \quad \forall n \geq n_0$

ie $n \leq 2^{c/2} \quad \forall n \geq n_0$, where $c$ is a constant

$\Rightarrow\!\Leftarrow$ $\therefore 2n \lg n \notin O(n)$

Contradiction

# Master Theorem

$T(n) = aT(\frac{n}{b}) + f(n)$ is defined on positive ints;

$f(n)$ is a positive-valued function $\forall$ pos ints $n$.

$a \geq 1$

$b > 1$

Then:

for some (small) positive value of $\varepsilon$

case 1: if $f(n) \in O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) \in \Theta(n^{\log_b a})$

case 2: if $f(n) \in \Theta(n^{\log_b a}) \Rightarrow T(n) \in \Theta(n^{\log_b a} \log n)$

case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$

and $\exists\, c,\ 0 < c < 1$ where $a \cdot f(\frac{n}{b}) < c \cdot f(n)$
(when $n$ is sufficiently large)

$\Rightarrow T(n) \in \Theta(f(n))$

regularity condition.

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \log n \qquad \underline{n^{\log_2 4} = n^2}$$

$$n^2 \log n \notin O(n^{2-\varepsilon})$$

$$n^2 \underline{\log n} \notin \Theta(n^2)$$

$$n^2 \log n \notin \Omega(n^{2+\varepsilon})$$

$\therefore$ MT is silent on this case.

$$T(n) = 4T\left(\frac{n}{3}\right) + n^2 \log n$$

$n^2$