Big O, Big Omega, Theta

Warm-up:

[Recall Thm: $\log_c n < n \quad \forall n \geq 2$
$\forall c \geq 2$]

Prove $3n^2 + 4n\log n + 5 \in O(n^2)$

Proof:

1. $3n^2 \leq 3n^2 \quad \forall n$
2. $4n\log n \leq 4n^2 \quad \forall n \geq 1$ ($\log_n \leq n \; \forall n$)
3. $5 \leq n^2 \quad \forall n \geq 3$
4. $3n^2 + 4n\log n + 5 \leq 8n^2 \quad \forall n \geq 3$
5. $3n^2 + 4n\log n + 5 \in O(n^2)$, as demoed by $c = \underline{8}$, $n_0 = \underline{3}$

$1 \leq n \quad \forall n \geq 1$
$n \leq n^2 \quad \forall n \geq 1$

Rule 1. Removal of constant factors

$c\,f(n) \in O(f(n)) \quad \forall$ constants $c > 0$

Rule 2. Transitivity of Big O

$f(n) \in O(g(n))$ AND $g(n) \in O(h(n))$
$\implies f(n) \in O(h(n))$

Proof: Suppose $f(n) \in O(g(n))$ and
$g(n) \in O(h(n))$.

Then $\exists c_1, n_1$ s.t. $f(n) \leq c_1 \cdot g(n)\ \forall n \geq n_1$

and $\exists c_2, n_2$ s.t. $g(n) \leq c_2 \cdot h(n)\ \forall n \geq n_2$.

$\Rightarrow f(n) \leq c_1 \cdot (c_2 \cdot h(n))\ \forall n \geq$

$\Rightarrow f(n) \leq (c_1 \cdot c_2) \cdot h(n)\ \forall n \geq$

$\Rightarrow f(n) \in O(h(n))$ as demonstrated by

$c =$ _____, $n_0 =$ _____. $\blacksquare$

Rule #3. Strange-but-true log domination rule.

$(\log n)^r \in O(n^s)\ \forall r, \forall s > 0.$

Eg $(\log n)^{100000} \in O(n^{0.00001})$

Rule #4 (a) degree of polynomial rule

Let $p(n)$ be a polynomial of degree k
with k^{th} coefficient > 0.

Then $p(n) \in O(n^k)$ and $n^k \in O(p(n))$

Proof: $p(n) = a_k n^k + \cdots \underbrace{a_i n^i}_{\leq |a_i| n^k} + \cdots + a_0 n^0$ $\forall n \geq 1$

$\therefore p(n) \leq \left(\sum_{i=1}^{k} |a_i|\right) \cdot n^k \quad \forall n \geq 1$

$\therefore p(n) \in O(n^k)$, demo'ed by $c = \sum_{i=1}^{k} |a_i|$ $n_0 = 1$

Exercise: Show $n^k \in O(p(n))$.

$\boxed{/\!/}$

Rule #4. Polynomial Rule

$p(n) \in O(q(n))$, when $p(n), q(n)$ are polynomials in n of degree k and t respectively, and $k \le t$

ai's and k, t are constants

Proof: Let $p(n) = a_k n^k + \dots + a_0 n^0$

$q(n) = b_t n^t + \dots + b_0 n^0$.

$a_k > 0$

$b_t > 0$

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \dots a_1 n^1 + a_0 n^0$$

$$\le a_k n^t + |a_{k-1}| n^t + \dots + |a_1| n^t + |a_0| n^t \quad \forall n \ge 1$$

$$= \left(\sum_{i=0}^{k} |a_i| \right) n^t \quad \forall n \ge 1$$

$$\le \left(\quad \right) b_t n^t \quad \forall n \ge 1$$

① $\Rightarrow p(n) \in O(b_t n^t)$

② Note: $b_t n^t \in O(b_t n^t + b_{t-1} n^{t-1} + \dots + b_0 n^0)$

by Rule "degree of polynomial"

$\Rightarrow p(n) \in O(q(n))$ by ①, ② and Transitivity of Big O. ▨

Rule #5. Product Rule

$$f_1(n) \in O(g_1(n)) \text{ and } f_2(n) \in O(g_2(n)) \implies f_1(n)\cdot f_2(n) \in O(g_1(n)\cdot g_2(n))$$

Proof: pretty easy. Uses, and is analogous to, the fact that

$$\frac{\begin{array}{c} a \leq b \\ c \leq d \end{array}}{a \cdot c \leq b \cdot d}$$

Left as an exercise for the student.

$$\frac{\begin{array}{lll} f_1(n) \leq c_1 \cdot g_1(n) & \forall n \geq n_1 \\ f_2(n) \leq c_2 \cdot g_2(n) & \forall n \geq n_2 \end{array}}{f_1(n)\cdot f_2(n) \leq c_1 \cdot c_2 \cdot g_1(n) \cdot g_2(n) \quad \forall n \geq \max(n_1, n_2)}$$

Rule #6. Log base is Irrelevant

$$\log_a n \in O(\log_b n) \quad \forall \overset{\text{constant}}{a, b} > 1$$

Proof: $\log_a n = \dfrac{\log_b n}{\log_b a}$ (Theorem)

$\therefore \log_a n = \underbrace{\dfrac{1}{\log_b a}}_{c} \cdot \log_b n$

$\therefore$ by constant factors rule

$$\log_a n = c \cdot \log_b n \in O(\log_b n) \quad \square$$

Rule #7. Reciprocal Rule

$$f(n) \in O(g(n)) \Rightarrow \frac{1}{g(n)} \in O\left(\frac{1}{f(n)}\right)$$

Proof: An exercise for the student.

Rule #8. Sum Rule

if $f_1(n) \in O(g(n))$ and $f_2(n) \in O(g(n))$

$$\Rightarrow f_1(n) + f_2(n) \in O(g(n))$$

Proof: An exercise for the student.

Rule #9 Less-Than Rule

$f(n) \le g(n) \; \forall n \ge n_0$ for some n_0

$\Rightarrow \; f(n) \in O(g(n))$

Proof: An exercise for the student.

A proof using the rules of Big-O

Claim: $3n^2 \log n - 160 \log^3 n \in O(n^2 \lg n)$

Proof:

		Rationale
1.	$\log n \in O(\lg n)$	L Base Irrevelant
2.	$3n^2 \in O(n^2)$	CF Rule
3.	$3n^2 \log n \in O(n^2 \lg n)$	1, 2, Product Rule
4.	$3n^2 \log n - 160 \log^3 n \in O(3n^2 \log n)$	By Less-Than Rule
5.	$3n^2 \log n - 160 \log^3 n \in O(n^2 \lg n)$	4, 3 Transitivity.

$\square$

When proving a Big O relation using the Rules

- refer to the rule explicitly by name in Rationale
- number the lines of your proof, and use the line numbers in your Rationale

Claim: $4n^2 - 1 \in O\left(\frac{3n^3}{\lg n}\right)$

Proof

0.	-1	$\in O(4n^2)$	Negative Rule
1	$4n^2$	$\in O(4n^2)$	Polynomial
1.5	$4n^2$	$\in O(4n^2-1)$	0, 1 Sum Rule.
2.	$\lg n$	$\in O(n)$	Log Domination.
3.	$\frac{1}{n}$	$\in O\left(\frac{1}{\lg n}\right)$	2, Recip Rule
4.	$4n^3$	$\in O(3n^3)$	CF Rule
5.	$\frac{4n^3}{n}$ ↖ $4n^2$	$\in O\left(\frac{3n^3}{\lg n}\right)$	4, 3 Product Rule.
6.	$4n^2-1$	$\in O\left(\frac{3n^3}{\lg n}\right)$	1.5, 5 Transitivity

Theorem: If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists, then

$$f(n) \in O(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c$$

for constant $c \geq 0$

if the limit exists

Proof:

$f(n) \in O(g(n)) \iff \exists\ c, n_0$ s.t. $f(n) \leq c \cdot g(n)\ \forall n \geq n_0$

$\iff \dfrac{f(n)}{g(n)} \leq c \quad \forall n \geq n_0$

$\iff \lim\limits_{n \to \infty} \dfrac{f(n)}{g(n)} \leq c$ $\blacksquare$

Def$^{\underline{n}}$ $\Omega(g(n))$

$f(n) \in \Omega(g(n))$ iff $\exists$ positive c, n_0 such that

$0 \leq c \cdot g(n) \leq f(n)\ \forall n \geq n_0$

let's just consider pos^{ve}-valued functions

Claim: $f(n) \in \Omega(g(n)) \iff g(n) \in O(f(n))$

Proof:

$f(n) \in \Omega(g(n)) \iff \exists\ c_1, n_1$ s.t. $c_1 \cdot g(n) \leq f(n)\ \forall n \geq n_1$

$\iff g(n) \leq \frac{1}{c_1} \cdot f(n) \quad \forall n \geq n_1$

$\iff g(n) \in O(f(n))$, as demo'ed

by $c = \frac{1}{c_1}$, $n_0 = n_1$ $\square$.

Def$^{\underline{n}}$: $\Theta(g(n))$

If $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$

then we say $f(n) \in \Theta(g(n))$.

Theta

Claim: $3n^2+1 \in \Theta(n^2)$

Proof: 1. $3n^2+1 \in O(n^2)$ Polynomial rule

2. $n^2 \in O(3n^2+1)$ Polynomial rule.

3. $3n^2+1 \in \Theta(n^2)$ 1, 2 Def$^{\underline{n}}$ Θ. $\blacksquare$