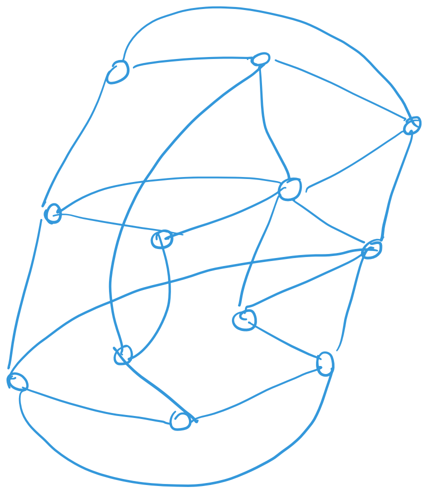
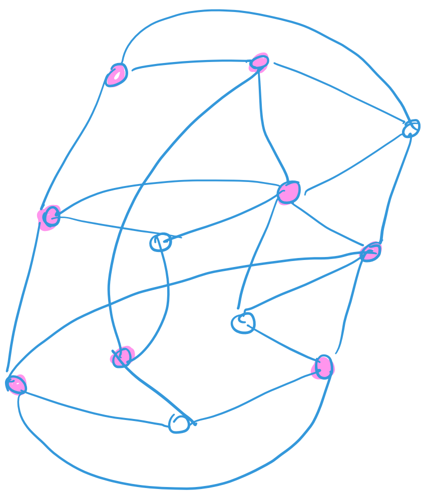
# 3-SAT $\leq_p$ Vertex Cover

Vertex Cover $= \{ \langle G, K \rangle \mid$ G is an undirected graph, and $\exists$ a set $X \subseteq V(G)$ where $|X| = K$ and all edges in G are incident with a vertex in $X \}$



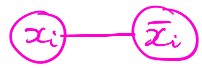
G:



$\langle G, 8 \rangle \in$ Vertex Cover

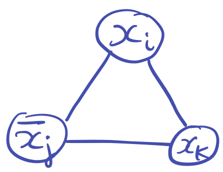
Theorem: Vertex Cover is NP-c

Proof: We reduce 3SAT to Vertex Cover as follows:

1. For each variable $x_i$ in $\phi$, create variable-gadget 



2. For each clause $C_i = (x_i \vee \bar{x}_j \vee x_k)$ create clause-gadget





3. Connect all clause-gadget $x_i$ with variable-gadget $x_i$

   clause-gadget $\bar{x}_i$ with variable-gadget $\bar{x}_i$

In the resulting graph, $\exists$ a vertex-cover of size

num variables + 2 · num clauses

iff $\Phi$ is satisfiable.

[To see that this is so, let's look at a small example. See below]

Claim: If $\Phi$ is satisfiable, then $\exists$ a vertex cover in $G$ of size num variables + 2 * num clauses

Proof: Let $T$ be a satisfying assignment of truth values to the variables of $\Phi$.

Start with VertexCoverSet $X = \emptyset$.

1. For each variable $x_i$, add to $X$ the vertex $x_i$ or $\bar{x}_i$ (depending on $T$) that is in $x_i$'s variable-gadget.
2. Since $\phi$ is satisfied by $T$, each clause-gadget has at least one edge out to a vertex that is in $X$.

3. Put the other two vertizes into $X$.

   As a consequence of this, every edge between a variable-gadget and a clause gadget is now covered by $X$.

4. Also, each variable-edge is covered, by step 1.
5. Also, every clause-edge is covered, by Step 3.

∴ all the edges of $G$ are covered by $X$, and

$|X|$ = num variables + 2 * num clauses. ∎

Claim: If $G$ has a Vertex Cover of size num variables + 2 * num clauses, then $\phi$ is satisfiable.

Proof: $ $G$ has such a cover $X$. Then:

1. For each variable-gadget, at least one of the two vertices are in $X$, since there is an edge there.
2. For each clause-gadget, at least 2 vertizes must be in $X$, since that is the only way to cover all edges in a △
3. No more vertices are in $X$, as that would exceed num variables + 2 * num clauses

∴ all clause-to-variable edges are covered by $X$

∴ since only 2 edges out from the clause to a variable are covered by $X$-members in the clause-gadget, the other must be covered by a variable gadget.

$\therefore$ Each clause has some literal that is made "TRUE" by the truth assignment that consists of all the variable-gadget vertices that are in $X$. $\blacksquare$