Last slide on P.L.

$\text{Prime}_a = \{ a^n \mid n \text{ is prime} \}$ 2, 3, 5, 7, 11, ...

Claim: Prime_a is not regular.

Proof: BWOC. $\S$ Prime_a is regular, and let p be its pump const.

Let t be the smallest prime $\geq K+2$.

Consider the string $w = a^t$. Note $w \in \text{Prime}_a$ and $|w| \geq p$. (in fact, $|w| \geq p+2$).

$w = a\,a\,\underbrace{a\,a\,a\,\cdots\,a}_{y}\,\underbrace{a\,a}_{\geq 2 \text{ extra } a\text{'s}}$ (p)

By PL, $w = xyz$ where $|y| > 0$, $|xy| \leq p$, and $xy^iz \in \text{Prime}_a$ $\forall\, i \geq 0$.

ie. $a^{|x|}\, a^{i\cdot|y|}\, a^{|z|} \in \text{Prime}_a$. $\forall\, i \geq 0$.

in particular when $i = |x| + |z|$, the result of pumping up is

$a^{|x|}\, a^{(|x|+|z|)|y|}\, a^{|z|}$

$= a^{(|x|+|z|)(|y|+1)}$ ... but $(|x|+|z|)(|y|+1)$ is not prime!

($|x|+|z| \geq 2$, $|y|+1 \geq 2$)

$\Longrightarrow\Longleftarrow$ $\therefore$ Prime_a is not regular. ▨

Closure Theorems

Recall that the class RL is closed under

- union
- concatenation
- $*$
- complement

$\Longrightarrow$ intersection

- letter substitution

$a \to 0110$
$b \to 1001$.

"every a is followed immed by b"
"every 0 is followed immed by 1"

We can use these facts to help find nicer proofs of Non regularity.

$$L = \{ a^i b^j \mid i, j \geq 0 \text{ and if } i = 1 \text{ then } j \text{ is prime} \}$$

Claim: L is not regular

Proof: BWOC. $\not\in$ L is regular. Let p denote L's pumping constant. Let q be smallest composite $q \geq p+2$

Consider the string $a^p b^q$?

Proof: If L is regular, then so is $L \cap L(ab^*) = L'$

Note that $L' = \{ ab^i \mid i \text{ is prime} \}$.

Known reg language

Claim: L' is not regular.

Proof: Bwoc. & L' is regular and has pumping constant p

Let r be the smallest prime bigger than or equal to $p+2$

Consider ab^r, which is in L' and $|ab^r| \geq p$ so by P.L.

$ab^r = xyz$ where 1. $xy^iz \in L'$, 2. $|y| > 0$ 3. $|xy| \leq p$.

So:

i) $y = ab^t$ for some $t \geq 0$. But then pumping down gets rid of all a's $\Rightarrow\Leftarrow$ (see L' defn).

or ii) $y = b^t$ for some $t \geq 1$.

:

use the Prime$_a$ proof.

$\Rightarrow\Leftarrow$

Whichever is the case for y, (i) or (ii), each leads to a contradiction.

$\Rightarrow\Leftarrow$

$\therefore$ L' is not regular.

Now suppose L is regular. Then by the closure theorems, so is $L \cap L(ab^*)$

i.e. so is L' $\Rightarrow\Leftarrow$

$\therefore$ L is not regular. ▨

Recalling from earlier slides that

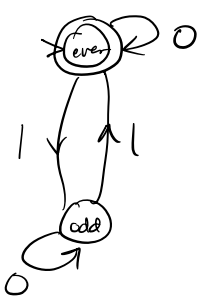
$$L_1 \in RL \text{ and } L_2 \in RL \Rightarrow L_1 \cup L_2 \in RL$$

Proof that if L_1 and L_2 are regular, so is $L_1 \cup L_2$

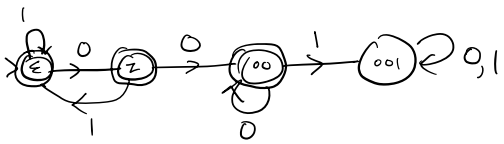
"Proof by example"

$\Sigma = \{0,1\}$

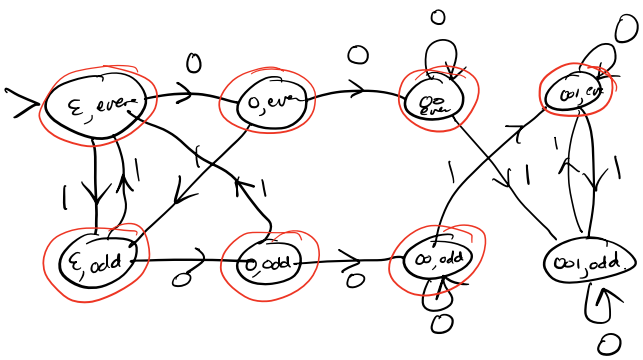
"$\#_1(w)$ is even"



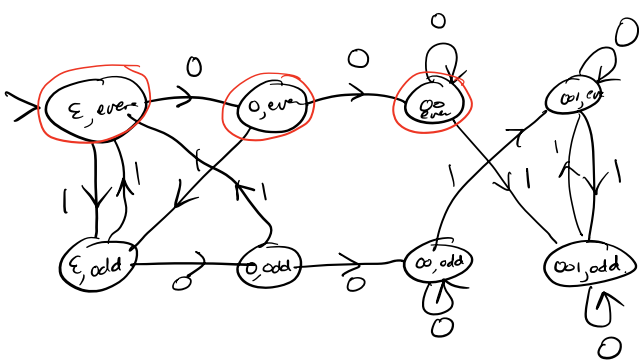
"has no 001 substring"



Union



Intersection



$L_{\neq} = \{ w \in \{a,b\}^* \mid \#_a(w) \neq \#_b(w) \}$

Claim: $L_{\neq}$ is not regular.

Proof: If $L_{\neq}$ were regular. Then $\overline{L_{\neq}}$ is also regular.

Then so is $\overline{L_{\neq}} \cap L(a^*b^*)$.

But this language is $A^n B^n$, which is known not to be regular.

$\Rightarrow\Leftarrow$

$\therefore$ $L_{\neq}$ is not regular.